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A TREATISE ON SURVEYING

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FOURTH EDITION

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CONTENTS

PREFACE TO THE FOURTH EDITION

This book, in its original form, was the work of many writers, each one a man whose name was a guarantee that what he had to say on the subject of Surveying was worthy of attention, but all writing more or less independently and without a very exact idea of the contributions to be expected from one another.

Under the circumstances it was inevitable that, though in many respects a distinct advance on pre-existing works on this subject, it should suffer from some defects due to the manner of its production.

It seemed desirable that an attempt should be made to remove these, as far as possible, by means of a general revision by a single writer. The present edition is the result of such an attempt.

The labour of revision has proved much greater than was anticipated. It seemed desirable to re-arrange the work so as to make not only a two-volume book on Surveying, but practically *two books*, whereof Part I should satisfy the requirements of students entering for the less difficult examinations in surveying, and of young engineers in countries already mapped; whilst Part II would supply the additional information required for the more searching examinations, or for work in unmapped districts, as well as serving as a useful introduction to the study of the higher geodesy.

This re-arrangement has made it practically necessary to re-write the greater part of the book.

But it can truly be said that it has been a labour of love, and, whilst by no means satisfied with the result, the writer hopes it will be found that some improvement has been effected.

He understands that the object of the original authors was to produce a book which should be truly educational. That is, it was to be neither a mere *cram* book, helping students to answer examination questions with the least possible trouble, nor a handbook enabling a practical engineer to perform this or that operation by mere rule-of-thumb. This object has the entire sympathy of the writer, and has been carefully kept in view.

The chapter on Instruments in Part I has been omitted, each instrument being described in its proper place. In Part II, the article on Sun-dials and the chapter on Gauging Streams and Rivers have been deleted to make room for added matter. The last-named chapter (chiefly the work of the late Professor Chadwick) was necessarily much abridged from its original form in order that it might ever be included in the book at all. It is hoped to re-issue it shortly, in the unabridged form, as a separate pamphlet. Some deletions have also been made in the chapter on Tidal Phenomena.

With these exceptions, it is hoped that nothing of special value in the original book has been omitted. The re-arrangement must speak for itself. The added matters are too numerous to receive individual mention, the following being a few of the more important: (*a*) The chapter on the History of Surveying, which has been specially written for this edition by Mr. E. H. Sprague, Assoc.M.Inst.C.E., Lecturer in Surveying at the Westminster Technical Institute. The writer feels that this chapter has added greatly to the value of the book, and he wishes to express his thanks for it to Mr. Sprague. (*b*) The method of least squares has been explained, and several worked examples are given showing its application to surveying problems. It was felt that this was necessary if the book was to serve as an introduction to the study of geodesy. (*c*) The adjustment of the errors in a triangulation has been treated much more fully than before; and the chapter on Base Line Measurements has been brought more nearly up to date. (*d*) The number of worked examples showing the application of the principles of surveying to engineering problems has been largely increased, especially in the chapters on Levelling and Traversing; whilst the chapters on Curve Ranging and Tacheometry have been very much amplified.

During the work of revision the writer has had the benefit of the advice and help of Mr. R. E. Middleton, one of the original authors. While the writer wishes to accept full responsibility for all faults, he hopes that the fact just mentioned has helped to reduce their number.

M. T. M. ORMSBY.

UNIVERSITY COLLEGE.
LONDON.

EXTRACTS FROM PREFACE TO THE FIRST EDITION

In the United Kingdom, and with Ordnance Maps to various scales, and more or less corrected to date, always at hand, the surveyor seldom, if ever requires to put into practice a knowledge of high-class, or Geodetic Surveying. The result is, that diplomas are granted to students who possess but a very limited knowledge of this class of work.

In the principal Colonies, however, these conditions do not obtain, and their Governments, not being satisfied with the limited acquirements of many English surveyors, insist on a local training, or apprenticeship and the possession of one of their own diplomas. This restriction has, in many cases, proved somewhat arbitrary.

About three years ago, a number of gentlemen interested in this question, and in the improvement of the standard of qualifications for English diplomas as Surveyors, met at the Surveyors' Institution by kind permission of the Secretary, and formed a committee to consider what steps might best be taken to secure a better position for the English student who might wish to seek employment in one of our Colonies.

The Council of the Surveyors' Institution were approached, and their Secretary, Mr. Julian Rogers, informed the committee that their members were prepared to look favourably on any efforts made in the direction indicated.

It was arranged that the committee should prepare, and submit a Text-Book, which the Council agreed to adopt, if satisfied with the same. The present Treatise on Surveying is the result.

The Authors beg to offer the above remarks as their justification for submitting this work to the verdict of public opinion, and whilst making no claim to having compiled a complete treatise on so large a subject as surveying, still it is hoped that all the information necessary to enable a student to acquire the knowledge required of a qualified surveyor is to be found in it.

The following is a list of the Contributors; and it is deeply to be regretted that the sections written by Mr. Leane (who died at sea when on his return from Africa) and by Major-General Woodthorpe, C.B. (appointed Surveyor-General of India just before his death) have had to be produced without the advantage to the Editor of being able to submit proofs for their revision:—

Reginald E. Middleton, M.Inst.C.E., M.I.Mech.E., F.S.I., Fellow of the Sanitary Institute.
Formerly Instructor in Surveying at the Central Institute of the City and Guilds of London, etc.

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A Consulting Engineer to the Crown Agents for the Colonies.

J. du T. Bogle, Colonel (late R.E.).
Formerly Instructor in Surveying, S.M.E., Chatham.

G. H. Leane, F.S.I.
Examiner in Surveying to the Surveyors' Institution, the Royal Agricultural Society, etc. (since deceased).

Major-General Woodthorpe, C.B. (late R.E.).
Appointed Surveyor-General of India just before his death.

SURVEYING

CHAPTER I

ADDITIONAL INSTRUMENTS

IN this chapter it is proposed to collect together particulars of some instruments of which every surveyor should have some knowledge, but which do not necessarily belong to any general method of surveying.

No attempt has been made to render the list exhaustive. Many "combination" instruments, for instance, will be found in the catalogues of the makers which are not here mentioned.

Some instruments which are practically obsolete have also been omitted, while instruments so well known as, for instance, ordinary drawing instruments have been taken for granted.

Some appliances are included which are "parts" rather than separate instruments.

Marquois Scales.—These scales seem to be very little used nowadays by draughtsmen, being regarded as chiefly useful for military drawings. For drawings of a limited size, a set of Marquois scales supplies the place of the set-square, the straight-edge, and the parallel ruler. The set consists of two scales of equal width, each a little over 1 foot in length (leaving about $\frac{1}{8}$ of an inch beyond each terminal graduation), and a set-square or triangle, the whole being made in stout boxwood of about $\frac{1}{5}$ of an inch in thickness.

The triangle has two of its sides of a length in the proportion of 3 to 1, the longest being the hypotenuse and the shortest the base. The remaining side is bevelled for use with the drawing-pen. There is an index in the centre of the longest side, which reads into the scales on the rulers.

On each of the edges of the pair of rulers, a pair of scales is inscribed, the inner one being divided into many parts of an inch,

from 20 to 60. The divisions of each outer scale are three times the length of the divisions of the corresponding inner scale. The latter are for use with the triangle, whose hypotenuse and base are in the ratio of 3 to 1, so that if the index above mentioned be moved along the divisions of the outer scale, the bevelled edge can be used for ruling lines at intervals apart, corresponding to the divisions on the inner or natural scale.

Scales.—Draughtsmen's scales are usually made about a foot long, of either boxwood or ivory—other materials, such as metal, vulcanite, celluloid, or paper, are sometimes used.

Both ivory and vulcanite expand and contract largely, the former with moisture, the latter with heat. A celluloid scale in the possession of the writer was about one per cent. out after a few years' use.

Paper scales are supposed to expand and contract with the drawing paper, but to this end it is best to construct a scale for each survey, and on each sheet of paper, as is done at the foot of each sheet of the Ordnance Survey maps.

The section of scales, which are usually divided on the bevelled edge, are either flat, oval, or triangular, an improved American pattern of the latter section consisting in hollowing out the centre portions and fining down the solid angles, clearness in the divisions being ensured by the use of a white enamel ground.

In this country surveyor's scales are usually divided to give (Gunter's) links on one side and feet (to the same scale) on the other.

The Opisometer.—The opisometer is an instrument for measuring the lengths of winding roads, paths, etc., on a map, and consists of a small mill-edged wheel revolving on a screw, which forms its axle. The axle is carried by guides fixed to an ivory or bone handle. The use of this little instrument is obvious, but it may be noted that it requires to be run back to zero over the scale of the plan to get the distance measured.

Reeves' Proportional Dividers.—The instrument consists of an ordinary pair of dividers, fitted with a movable scale, by means of which proportional measurements can be made, lines divided into any number of parts, and latitudes and longitudes accurately read from maps and charts. It will serve as a diagonal scale or vernier for exact measurements of various kinds, etc.

For particulars apply to the makers, Messrs. C. F. Casella & Co., Whitehall, London.

The Box Sextant.—This instrument is somewhat similar in construction to the optical square, but whereas in the latter instrument both mirrors are fixed, and only angles of 90° can be measured,

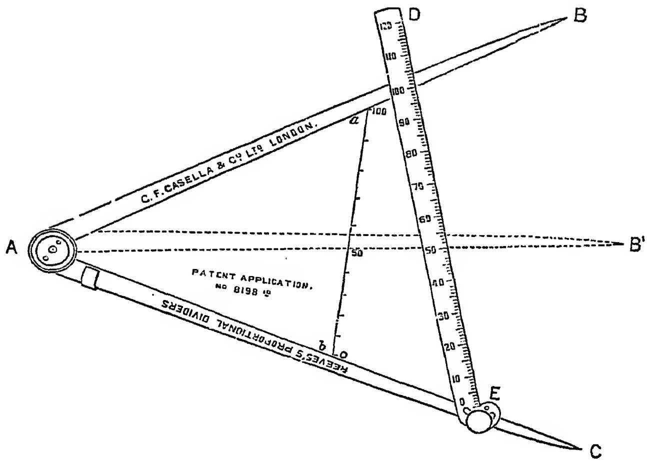
D
B
A
B'
E
C
a
b
C. F. CASELLA & Co LTD LONDON.
PATENT APPLICATION.
No 8198 10
REEVES'S PROPORTIONAL DIVIDERS

FIG. 1.

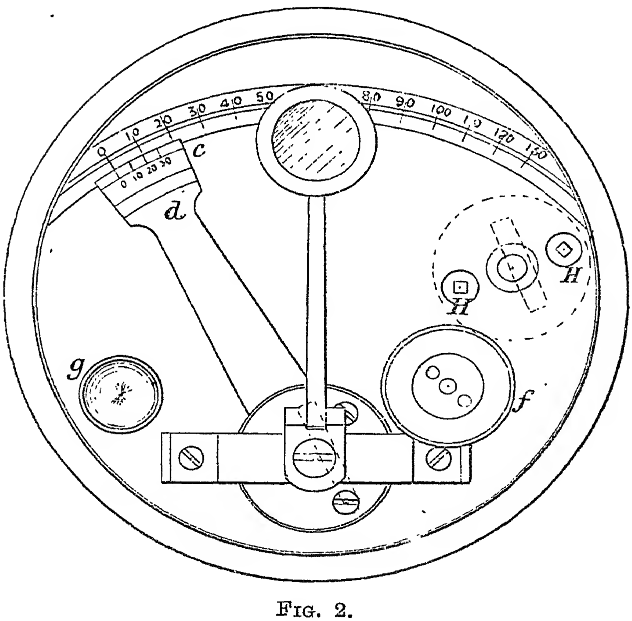
c
d
g
f
H
H

FIG. 2.

in the sextant one mirror only is fixed, and the angle made between it and the second mirror can be altered so that any angle from 0° to 110° or thereabouts can be measured by the observer. Fig. 2 is a plan of the top, and Fig. 3 a sectional plan.

The instrument consists of a cylindrical metal box about 3 inches in diameter and $\frac{3}{4}$ of an inch thick, covered when not in use by a cylindrical case which screws on to the lower edge of the

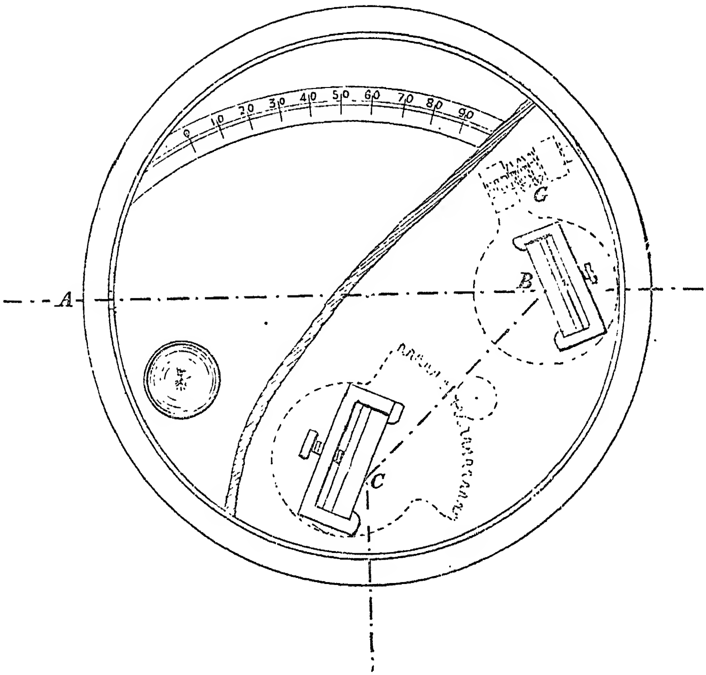

Fig. 3.

instrument, but which, when the instrument is in use, is reversed and screwed on at the back, serving as a handle.

The top of the box, when in use, is formed of a brass plate about $\frac{1}{8}$ of an inch thick, to which the various mirrors, etc., to be described are attached.

In the side, at A, there is an eye-hole pierced in a slide, and in some instruments a small telescope is provided. Opposite to

this eye-hole, and on the other side of the box, about $\frac{1}{3}$ of the circumference is cut away.

Between the eye-hole A and this opening the mirror B is fixed, and is called the "Horizon Glass." The lower half of this glass is plain, whilst the upper half is silvered.

In the sextant, the second mirror, or "Index Glass" C, is so arranged that it can be caused to revolve on its axis by means of a rack to which it is attached, and a toothed wheel which engages in the same. The toothed wheel is connected to the milled head *f* which appears outside the case (Fig. 2).

An arrangement is provided by which dark glasses may be interposed between the eye and the mirrors to enable the sun to be observed with safety.

Angles greater than about 110° cannot be observed with the box sextant. Blakesley's "wide angle" sextant referred to in Chap. V. Vol. I., enables greater angles to be measured. Many makers supply modifications of the box sextant for the same purpose, some of which, for instance, will be found described by Mr. Stanley in his "Surveying Instruments" (Messrs. E. & F. N. Spon).

Principle of the Sextant.—Referring to Fig. 4, let YX be the index glass, UV the horizon glass, A the position of the sighting aperture or telescope.

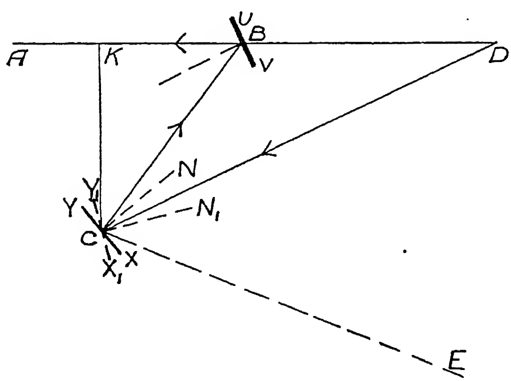

FIG. 4.

Then let any object D be viewed through the unsilvered part of the horizon glass, and let the index glass be turned into such a position, XY, that the object D, as seen by double reflection,

appears to coincide with itself as seen directly. The path of the reflected rays is shown by arrows in the figure.

Let the vernier attached to the index arm be then read.

Now let E be any other object, and let the index glass be turned to the position X_1Y_1, such that the reflected image of E now appears to coincide with D as seen through the horizon glass. That is the ray EC, after reflection at C, now follows the same path CB as was previously followed by the ray DC. Let the vernier be again read.

Let CN, CN_1 be the normals to the mirrors in the two positions.

$$\begin{aligned} \text{Then } BCE &= 2 \times BCN_1 \\ BCD &= 2 \times BCN \\ \therefore \text{ by subtraction, } DCE &= 2 \times NCN_1 \end{aligned}$$

But NCN_1 is the angle through which the index glass has turned, and this is measured by the difference between the two vernier readings. Hence the required value of DCE is known.

In order that the angle DCE may be obtained directly, without the trouble of doubling the angle NCN_1, each *half degree* on the arc is numbered as *one degree*.

It is clear that, to obtain the exact value of any angle with the sextant the *centre of the index glass* should be exactly over the station C, where the angle DCE is required, and the work should be carried out as here described.

If the objects are distant, however (*how far* will be considered later), it may be taken that when D coincides with itself the ray DC will be practically parallel to DA.

In this case it is very easy to show that the index and horizon glasses must be parallel in the first position.

Hence if the vernier be so set as to read zero when the glasses are parallel, it is clear that we need only bring E into coincidence with D, and the reading then will give at once the value of the angle DCE.

It is the aim of the makers to arrange matters in this way, but the adjustment is continually varying with temperature, etc. Any discrepancy from zero in the readings, when a distant object coincides with itself, is called the "index error," and must be tested frequently.

As this reading may fall on either side of zero, sextant arcs are always inscribed with a few degrees on the other side of that point. This small arc is called the "arc of excess."

Readings on that part of the scale are said to be "off the arc" or "on arc of excess."

There is usually only one set of figures on the vernier, however. Hence a subtraction sum must be performed to get the true reading on the arc of excess, because the figures there run the *opposite way* to those on the main scale, and the vernier figures should therefore run the opposite way also.

Thus with a main scale divided to half degrees, and thirty divisions on the vernier, giving minutes, if the arrow lie between zero and the next division off the arc, and coincidence takes place at number 19 on the vernier, the index error is 11 minutes *off the arc.*

For this reason the writer would much prefer to see astronomical sextants, in particular, with zero much farther back, so that no index error would ever be off the arc.

The index error is to be *added* (if *off* the arc) to all angles read by simply bringing the two objects into coincidence. It must be *subtracted* if *on* the arc.

If the objects are near, however, the work must be carried out as described at first. We will now consider at what distance DC may be regarded as parallel to DA.

We will suppose that the vernier reads to minutes, and that, in consequence, we consider that we may neglect errors less than 30″.

We shall further suppose, referring to Fig. 4, that in the particular instrument BC = 1·5 ins., CBA = 45°. Then the problem is to find the distance of D so that the angle BDC may not exceed 30″.

$$\text{Now } CD : BC :: \sin DBC : \sin BDC.$$

$$\therefore \quad CD = \frac{BC \sin DBC}{\sin BDC} = \frac{1{\cdot}5 \times \sin 45^\circ}{\sin 30''} \text{ inches}$$

$$= 600 \text{ feet, about}$$

This figure will vary with the dimensions of the particular sextant, of course; and it may be that if the angles are to be laid off on an ordinary protractor or station pointer we may consider say 3′ a negligible error (in which case, with the above dimensions, the distance CD becomes about 100 feet), but clearly we must not expect the best result if we ignore the preliminary reading on a near object.

Disadvantage of the Sextant.—An inherent defect of the sextant, for ordinary surveying purposes, is that it necessarily gives the angle in the plane containing the observer's eye and the two points observed.

If the ground slopes at all, this plane is necessarily inclined, so that the angle obtained is *not* the true horizontal angle. If the angles of elevation from the observer to each point be also measured, it is possible to calculate the true value of the angle.

The reduction is best performed by the aid of spherical trigonometry. It is clearly independent of the length of the rays, so we make CD = CE. CZ is vertical (Fig. 5), and we have measured the angle DCE = θ, and the angles α, β, of elevation of E and D respectively. Now in the spherical triangle ZDE, the angle Z gives the angle between the vertical planes through D and E, and hence the required horizontal angle.

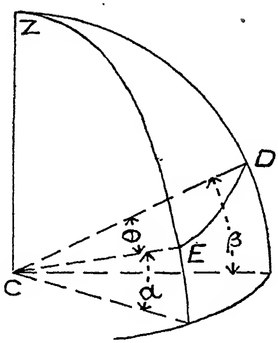

FIG. 5.

The side DE = measured angle θ

" ZE = $90° - \alpha$

" ZD = $90° - \beta$

We can solve the triangle from these sides to find the angle Z.

If either α or β is an angle of depression, it must be reckoned *minus*.

The logarithmic formula for solving the spherical triangle ZDE is

$$\tan \tfrac{1}{2}Z = \sqrt{\frac{\sin (s - ZD) . \sin (s - ZE)}{\sin s . \sin (s - DE)}}$$

where $s = \frac{1}{2}(DE + ZD + ZE)$.

The following formula is taken from the "Course of Mathematics," at the Royal Military Academy, vol. iii., and is derived from the above.

Let α, β be the angles of elevation in seconds, θ the observed angle, and x the correction to be applied to it in seconds, then—

$$x = \left\{\left(\frac{\alpha + \beta}{2}\right)^2 \tan \frac{\theta}{2} - \left(\frac{\alpha - \beta}{2}\right)^2 \cot \frac{\theta}{2}\right\} \sin 1''$$

α and β are assumed to be small angles.

If $\alpha = 10°$, $\beta = 10°$, $\theta = 60°$,

the first formula gives $Z = 56° \ 51' \ 48''$

and the second gives $x = - 3° \ 1' \ 20''$

$\therefore Z = 56° \ 58' \ 40''$

If $\alpha = 6°$, $\beta = 1°$, $\theta = 60°$,

then $Z = 59° \ 56' \ 2''$

$x = - 3' \ 56''$

which results differ by only 2″.

The student should check these results for himself.

Adjustment of the Box Sextant.—The adjustments of the box sextant are two in number:—

(1) To make the plane of the horizon glass parallel to the axis of the index glass.

Look at any vertical object such as the corner of a house, with the plane of the sextant horizontal, and turn the index glass until the two images coincide. Then slowly rotate the plane of the sextant round the line of sight as axis to bring it round to a nearly vertical position, looking all the while at the object. If it continues to look like *one* line, the adjustment is good; but if it separates into two parallel images, a key (g, in Fig. 2) will be found in the top of the box, which fits on to the square-headed screw H, just above the horizon glass. The latter is adjusted by this key to bring the images back to coincidence. Repeat till perfect.

This adjustment may also be made as described later (p. 10) for the nautical sextant.

(2) To correct the index error, if any.

Look at any very distant point not nearer than 1000 feet with the vernier exactly set to zero. If only one image is visible, the adjustment is good; but if two are seen, a square-headed screw (G, Fig. 3) will be found in the side of the box, at the back of the horizon glass. This is adjusted by the same key as before, to bring the images to coincidence.

Double Sextants.—Sometimes two sextants, back to back, are enclosed in the same box, so that they can be used by one observer to measure two angles (from a common centre point) at the same time. They are especially useful in hydrographic work for fixing the position of a sounding by observations from the boat to three known points.

Hadley's 8-inch Sextant.—This instrument is similar in principles of construction to the "box sextant" described above. It is well adapted to the purposes of the explorer, being portable and easy of manipulation, though, from being held in the hand, the extreme accuracy of fixed instruments is not to be expected of it.

The 8-inch sextant, reading to 10" of arc, is probably the best size, though the 6-inch, also reading to 10", is preferred by some on account of increased portability.

Stands are provided for use with sextants. These consist of a tribrach with an upright pillar, on the top of which is a sort of universal joint, carrying a transverse arm, to one end of which the sextant is screwed, the opposite end carrying a weight, balancing the sextant. With this stand, great steadiness may be obtained. The weight of the sextant and stand, however, is not much less than that of a theodolite, on the whole the more convenient instrument.

These sextants differ from the box sextant in that they are usually of larger size, and are of more open construction, not being enclosed in a box at all.

The index arm, which carries the index glass, is fitted with clamping and tangent screws like a theodolite, and there are usually two telescopes provided, one low-power erecting telescope, and an inverting one of higher power, the latter usually fitted with two parallel cross hairs, or sometimes four, forming a square.

As it is the instrument chiefly used for the astronomical determination of position at sea, it is frequently called the *nautical sextant.*

The telescopes are screwed into a ring which can be moved, by a special key, in a direction perpendicular to the frame of the sextant, so as to equalize the brightness of the images in observing stars.

Dark glasses of different densities are provided over each glass.

Sextants of this kind are sometimes made specially for sounding work, when all astronomical adjuncts are omitted.

Adjustments of the 8-inch Sextant.—The following are the adjustments of the 8-inch Hadley's sextant:—

1. Index glass for perpendicularity.
2. Horizon glass for perpendicularity.
3. Parallelism of sight-line of telescope to plane of sextant.
4. Index correction.

1. *Index Glass for Perpendicularity.*—This adjustment is effected when the silvered surface of the glass is perpendicular to the plane of the sextant. Set the index arm near the middle of the arc, then, placing the eye near the index glass, observe whether the arc seen directly, and its reflected image in the glass, appear to form one continuous arc, which will be the case *only* when this adjustment is perfect. The index glass can be set up by loosening the small screws which fix the plate that carries the glass, and wedging it forwards or backwards with a piece of paper as found necessary.

2. *Horizon Glass for Perpendicularity.*—Having completed the first adjustment, direct the telescope to a star, and move the index arm until the reflected image of the star appears to pass the direct image. If one image passes on either *side* of the other, the horizon glass is not parallel to the index glass, and must be adjusted by means of the small screw provided for that purpose, until one image passes over the other. For the above adjustment, a star of the third magnitude will afford greater precision than the brighter ones. Any distant well-defined terrestrial object may be substituted for the star, though the latter is to be preferred when the sextant is required for astronomical observations.

Many observers prefer to adjust their sextants so that one image passes just clear of the other.

3. *Parallelism of Sight-line of Telescope to Plane of Sextant.*—Turn the sliding tube of the telescope round till two of the wires are parallel to the plane of the sextant. Select two stars, or the moon and a star, not less than 90° apart, and bring them into contact on the right-hand wire. Then keeping the index clamped, change the position of the sextant till they appear on the other wire. If the contact remain perfect the line of sight midway between the wires is parallel to the plane of the instrument, and *vice versâ*. The adjustment of the telescope is effected by means of two small opposing screws in the ring which carries it.

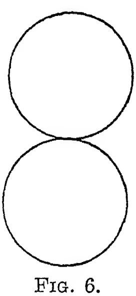

Fig. 6.

If the objects appear to separate on the wire *further* from the index arm the object glass end of the telescope droops towards it, and *vice versâ*.

4. *Index correction.*—As the index correction for a sextant is continually varying with the temperature, it is necessary to determine it both *before* and *after* each set of observations. The method of procedure is as follows:—

(*a*) *With the Sun.*—Clamp the index at about 30 minutes from zero on the arc, look towards the sun, and turn the tangent screw until the two images *just touch* one another, as shown in Fig. 6. This is called "making contact," and is the method always used in observing the sun, because the centre is not marked in any way, and it would be exceedingly difficult to say when the two discs *exactly* covered one another. The reading of the vernier should now give the angular distance of the sun's upper edge (or *limb*, as it is called) from the lower limb, which should be equal to the angular diameter of the sun as

given in the Nautical Almanac. Now note the reading of the vernier, say 31′ 56″. Then clamp the index to about 30 minutes from zero on the arc of excess, and as before, perfect the contact of the two images. Note the reading of the vernier (say 31′ 22″). Half the difference of these two readings is the index correction, *additive* when the reading *off* the arc (*i.e.* on the arc of excess) is the greater of the two, and *vice versâ.*

Example.

Reading on arc = 31′ 56″
,, off arc = 31′ 22″

Difference = 34″
Index correction *subtractive* = 17″

This observation may be checked by comparing *one-fourth of the sum of the two readings* with the sun's semi-diameter for the day given in page ii. of the month in the Nautical Almanac. In the above example the sun's semi-diameter by observation would be 15′ 49″·5.

(*b*) *With a Star.*—Bring the direct and reflected images of the star into coincidence, and read the arc. This reading is the index correction, *additive* if *off* the arc (*i.e.* on the arc of excess), and *subtractive* if *on* the arc.

Errors of Eccentricity and Graduation.—Besides index error, sextants are subject to errors of eccentricity and graduation. The first arises from the centres of the arc and the index arm not being truly identical. The observer may eliminate its effects by a judicious pairing of observations, *i.e.* when observing meridional stars for latitude, choose pairs of stars at about equal distances north and south of the zenith, and similarly, when observing ex-meridional stars for time or azimuth, choose pairs at about equal altitudes east and west of the meridian, adopting in each case the mean of a pair of results for a single determination.

The *graduation* of a sextant may be measured by comparing the coincidence of the ends of the vernier with the graduation of the scale at different parts of the arc.

The sextant is *par excellence* an instrument suited to nautical requirements, and is of the greatest value in making maritime surveys of coasts, harbours, estuaries, etc., and wherever soundings have to be taken and mapped.

Altitudes of Heavenly Bodies.—The altitudes of heavenly bodies are measured with the sextant at sea, from the "sea horizon," but on land it is necessary to use an "artificial horizon," of which there are many different forms on the market.

Artificial Horizon.—The angle observed with the "artificial horizon" is however double the altitude of the body observed, as is evident from an inspection of Fig. 7. The mercurial artificial horizon is undoubtedly the best form. It consists, in its most common form, of a tray 5 inches by $2\frac{5}{8}$ inches internal dimensions, into which the mercury is poured from a metal bottle. It is covered by a glass roof-like shade to protect the surface from agitation in windy weather. This shade must be made of plate glass, with truly parallel face, or the rays passing through them will be unequally refracted. Test this by taking some observations,

first *with* and then *without* the cover, or with the cover reversed end for end. Any error from this source may be more or less eliminated by reversing the cover in this way for half the observations.

Instructions for Observing Altitudes of Heavenly Bodies with the Sextant.—The following instructions for observing with the sextant should be followed :—

1. Focus the telescope before you put it in the socket.

2. Place the wires of the telescope parallel and perpendicular to the plane of the instrument.

3. When observing the angle between two objects of different brightness, always, if possible, direct the telescope to the fainter.

4. When observing the altitude of the sun, use the dark glass on the eye-piece in preference to the movable dark glasses, in order

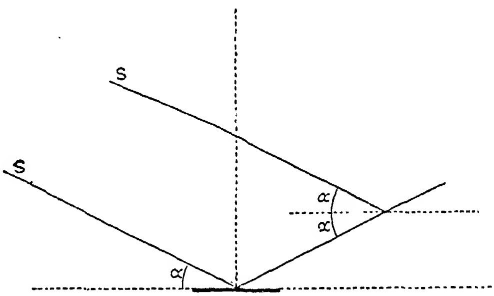

FIG. 7.

to avoid errors arising from the possible non-parallelism of the surfaces of the latter.

5. When observing the altitude of the sun, take the mean of alternate observations (or sets of observations) of the upper and lower limb, in preference to applying the correction for semi-diameter.

The reason of this is that different observers do not take "contacts" in precisely the same way, some taking it at moments when the contact is more fully developed than others. If, however, the same observer takes *both* upper and lower limbs in the same manner, the mean of the two observations will be correct, whereas the application of the semi-diameter value to one observation might not give him so accurate a result.

6. Care must be taken in observing the sun to be certain which limb is being observed. In Fig. 8, AH is the artificial horizon.

The full lines show the paths of rays from the upper and lower limbs of the sun, as reflected from AH, while the dotted lines show the paths of rays reflected from the index and horizon glasses. The telescope is supposed to be of the inverting type. To the eye of the observer, the upper limb *appears* uppermost in the image formed by the full rays (*i.e.* the image in the artificial horizon), and *lowermost* in the image formed by the sextant glasses. Thus if we make contact so that the former image is uppermost, we shall be observing the altitude of the sun's *lower* limb, and *vice versâ*. In paired observations, it is really only necessary to be sure of reversing the images. If there is no dark glass on the telescope, we may ensure

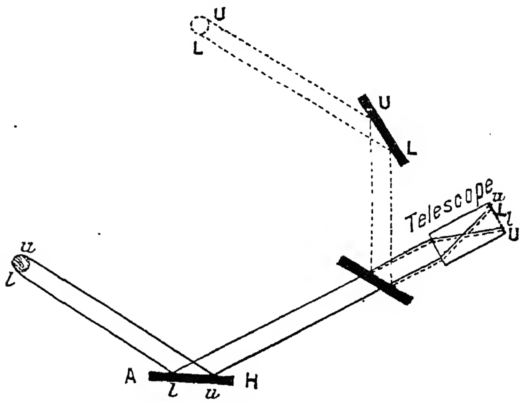

Fig. 8.

this by using different-coloured glasses in front of index and horizon glasses.

7. To "bring down" a heavenly body, arrange yourself in front of the artificial horizon in such a position that when the sextant is brought down to the position for observing, your eye will be properly situated for seeing the image through the telescope in the artificial horizon. Set the sextant (without clamping) to zero, and direct the telescope towards the heavenly body, so that you can see both images in the field of view, then, holding the index arm in the left hand, keep the reflected image always in the field of view, whilst you turn down the telescope with the right hand till it points to the artificial horizon, when the direct image will be readily found there.

It is best, if the body brought down be a star, to bring it down without the telescope. When the images are nearly together,

clamp, quickly insert the telescope, bring the sextant approximately to the correct plane, and give it a turning motion (by wrist action) round the axis of the telescope, taking care that the latter is aimed so that the image in the mercury is constantly visible. The second image should then be easily picked up, and exact contact made. If the angle is quickly read, it is generally unnecessary to bring the body down again for the next observation.

8. Observations for index correction should be taken *before* and *after* using the sextant for taking an observation.

The Six-inch Transit Theodolite.—A general description of the transit theodolite has been given in Chapter IV., Part I., so that it is only necessary to treat here of its use as an instrument for taking astronomical observations.

For the above purpose the 6-inch is the most suitable size, though the 5-inch lighter instrument is sometimes preferred.

The axis of the instrument is usually perforated, and a lamp supplied wherewith to illuminate the field of view, but if not perforated, the field can be illuminated by directing the rays of a lamp on to a strip of paper, or a metal reflector, fixed to the object end of the telescope, and bent to an angle of 45° with the axis.

When taking a "set" of observations, the telescope should be reversed between each pair of readings, in order to eliminate any residual error after adjusting for "collimation." (Face right and face left, *vide* Part I., p. 237.)

When observing the sun—

(*a*) A dark glass must be used over the eyepiece.

(*b*) Accurate observations of both limbs must be taken, leading and following limbs for azimuth, upper and lower limbs for altitude.

(*c*) Observation is made by the contact of a limb of the sun with the centre vertical or horizontal wire.

The surveyor may have to use a theodolite unprovided with a dark glass for the eyepiece. In such a case, one of the writers has successfully resorted to the following expedient for sun observations. A disc of cardboard was fitted inside the dew-cap. In the exact centre of this disc a small hole was pierced, and carefully reamed out with a pencil cut to a pyramidal form. With a 5-inch theodolite a hole rather less than one-eighth of an inch in diameter gave an image of the sun which could be comfortably observed. If this device is used, precept (*b*) is especially important on account of "aberration."

When reading the vertical circle, index errors are eliminated by reversing the telescope between each pair of readings, and

taking half the readings with "circle (or face) right" and half with "circle (or face) left."

It is not practically advisable to expend time in bringing the vertical circle bubble to the exact centre of its run for each observation, but it is sufficient if both ends of the bubble are within the scale marked on the tube, so that we can read them.

Level Correction.—The upper level is unquestionably more convenient for astronomical work if on the T piece than on the telescope. Having levelled up for vertical angles (see Part I., p. 241), the bubble will not *remain* exactly central in all positions in consequence of the compression of the ground, etc. We therefore read both ends of the bubble on its graduated scale, and apply a "level correction" accordingly to the observed angle.

The reading of the bubble is registered as "object-end" O, or "eye-end" E, according as the end is near the object-glass or eyepiece. If the O readings be greater than the E, the correction is *added* to the altitude on the vertical arc, and *vice versâ*.

Let L = level correction.

n = no. of observations.

V = value of one division of the scale (ascertained specially for each instrument).

Then
$$L = \frac{O - E}{2n} \times V$$

Here O and E stand for the sums of the readings of the object-glass and eyepiece ends of the bubble respectively. It is assumed that several observations have been taken, and that it is proposed to calculate from the mean. The formula gives the *mean value* of the level correction for all the observations.

Otherwise, the correction for each observation must be worked out separately, putting $n = 1$.

Example.-

O Readings.	E Readings.
14	15
$13\frac{1}{2}$	$15\frac{1}{2}$
17	12
15	14
$59\frac{1}{2}$	$56\frac{1}{2}$

$$\text{Mean excess of O readings over E readings} = \frac{3 \text{ graduations}}{4}$$

$$\text{Value of graduation of bubble, say} = 12''$$

$$\text{Correction for bubble readings therefore} = +\,(12'' \times \tfrac{3}{8}) = 4''{\cdot}5$$

To find Value of One Division.—The angular value of 1 division of the bubble is usually marked on the tube by the makers; but, in any case, it can be determined as follows.

Direct the telescope on a distant but well-defined object. Clamp the vertical circle, and read both ends of the bubble and the vertical circle. By *one of the foot-screws*, elevate or depress the telescope until the bubble has travelled over a given number of divisions. Direct the telescope again on the object by means of the vertical circle tangent screw, read the vertical circle, and again read both ends of the bubble.

The difference between the readings of either end of the bubble tells the number of divisions moved by the latter (the results should very nearly agree), and the difference between the readings of the vertical circle tells the corresponding angular value.

Example.—

	O	E	Vertical circle.
1st reading . . .	5·5	14·5	1° 40′ 49″
2nd „ . . .	17·5	2·0	1° 43′ 53″
∴ Difference . . .	12	12·5	0° 3′ 4″

$$\therefore \text{(Mean) } 12{\cdot}25 \times V = 3'\ 4''$$

$$\therefore V = \frac{3'\ 4''}{12{\cdot}25} = 15''{\cdot}02$$

The result should be pretty constant, no matter how many divisions be used in the test.

The value of 1 division of the bubble is liable to considerable variation with temperature. In the Ordnance Survey tests the value of 1 division of the bubble on the 3-foot theodolite varied from 1 second at 66° to about 5 seconds at 32°.

Detached Level-tubes.—If the level-tube to be tested is not attached to the theodolite, it can be tested either by attaching to the vertical circle of any theodolite, or by means of a "level-trier," or "level-tester," such as used by instrument makers and others (see Stanley's "Surveying Instruments").

Foundation for Theodolite.—It is most essential that the theodolite should be supported on a very solid foundation. The nature of the ground affects the method of doing this. Generally, it is sufficient to drive strong stakes as far as possible into the earth, and then to cut them off level with the surface, to form an immediate support for the stand of the instrument.

For large surveys of the best class, elaborate precautions must be taken to ensure steadiness. If a scaffold is necessary, separate scaffolds must be erected for the instrument and for the observers.

Theodolite and Sextant compared.—A transit theodolite

possesses the following advantages over the sextant for astronomical observations.

(1) It measures horizontal angles directly, and a round of several angles can be measured with less trouble than with a sextant.

(2) It measures small vertical angles of elevation or depression, which could not be measured with an artificial horizon.

(3) Its telescopic power is usually higher than that of the sextant.

(4) It can be manipulated so as to eliminate instrumental errors such as eccentricity, collimation, and index errors.

The disadvantages of the theodolite are to be found in its greater cost, bulk, and weight. For astronomical observations, where altitudes only have to be observed, the sextant is to be preferred to a small theodolite with 3 or 4 inch circle.

Timekeepers.—For work in the field, a keyless lever watch or semi-chronometer is recommended, such watches being now easily repaired, and less liable to get out of order, than full chronometer watches. They can be carried in the pocket, under conditions of fairly rough usage, and will not fall far short of a chronometer, at least in the regularity of their rates. For convenience in counting, a watch should not beat more than five times in two seconds.

For astronomical purposes, it is necessary to know the error and rate of each clock, chronometer, or watch at any given date, from which the chrometer correction at any other date can easily be found.

The Error of a Watch.—It is well to mention here that "the error of a watch" is the amount which it is fast or slow on correct time, and hence a watch *fast* has a (+) error and a watch *slow* has a (−) error.

"The correction to watch time" is the amount to be applied as a correction to the time indicated by a watch to reduce the same to correct time. Therefore

A watch *fast* has a (−) correction
A watch *slow* has a (+) correction

It is usual, however, in astronomy, and in surveying generally, to speak of this correction as the "error." Thus, if we say that a watch has a *minus* error, we really mean that the *correction* is to be subtracted, or that the watch is fast.

In the same way we speak of the "index error" of a sextant when we really mean the index *correction*, the latter being, of course, what is practically required.

Other Forms of Theodolite.—The transit theodolite is the form now almost invariably used in all kinds of surveying work.

The two forms next best known are the Y and Everest theodolites. For full particulars of these, the reader is referred to Stanley's "Surveying Instruments." A few notes about the Y instrument are, however, appended.

The "Y" Theodolite.—The "Y" theodolite is similar in construction to the transit as far as concerns the stand and work below the "A" side supports. The vertical arc is little more than half a complete circle, the balance being cut off and replaced by a flat plate, carrying at either end a bearing or "Y" to carry the telescope, from which the instrument takes its name. The telescope, which is similar in its optical arrangements to that of the transit, is provided with two turned collars, which fit the bearings, so that the tube may be revolved on its longitudinal axis, or when the caps of the bearings are removed, the telescope may be lifted out and replaced end for end. The spirit-level is fixed to the telescope so as to be underneath, and out of the way, when in position for use. No adjustment is provided in the bearing of the horizontal axis, as in the case of the transit, this adjustment not being of such importance, since the telescope cannot be used to observe angles of great elevation or depression. Only one vernier is fitted to the vertical arc, fixed to one side support, and in some instruments only one level is provided on the upper plate, at right angles to the telescope, levelling in the other direction being accomplished by setting the vertical arc to zero and using the level attached to the telescope or by turning the plate through 90°. A clamp and tangent screw are provided to the vertical arc. The "A" side supports are made lower than in the case of the transit, as there is no need to provide room for transiting the telescope.

The axis of the telescope Y's is supposed to be set by the makers permanently at right angles to the horizontal axis, and the latter at right angles to the vertical axis, so that no adjustment is provided for these.

Adjustment of the "Y" Theodolite.—

1. Collimation.
2. Telescope level.
3. Vertical axis of rotation.
4. Vertical circle or "zero of altitude."

1. *Adjustment for Collimation.*—Turn the telescope by a rotary motion in the Y's until one pair of diaphragm screws is vertical, and sight the cross-wires on some well-defined object. Now turn the telescope half round in the Y's, and if the intersection of the wires has moved *vertically* from the object, bring it back *half* by the vertical tangent screws, and *half* by the *now* vertical pair of diaphragm screws. Now rotate the telescope through 90°, and repeat the process with the other pair of diaphragm screws, *now* vertical. Repeat until the intersection of the cross-wires remains

fixed on an object during the whole of a revolution of the telescope in the Y's.

2. *Adjustment of Telescope Level, i.e.* to make the telescope level parallel to the axis of revolution of the telescope in the Y's, and therefore parallel to the line of collimation. Bring the bubble of the telescope level to the centre of its run and clamp the vertical arc. Reverse the telescope end for end in the Y's, and if the bubble does not still remain in the centre, correct *half* the displacement by the vertical tangent screws, and *half* by the screw at one end of the level. Now rotate the level *slightly* in the Y's, and if the bubble moves, place the level in a truly vertical plane by means of the lateral screws provided at the other end of the level. Repeat till perfect.

3. *Adjustment of Vertical Axis of Rotation, i.e.* to make the vertical axis of rotation truly vertical, and to set the line of collimation and the levels of the "horizontal plate" at right angles to it. Turn the telescope till it is exactly over two opposite foot-screws, and bring the bubble of the telescope level to the centre of its run, by the vertical arc tangent screw. Now clamp the "lower plate," and turn the "upper plate" through 180° till the telescope is again over the same pair of foot-screws. If the bubble be not still in the centre of its run, correct *half* the displacement by the vertical tangent screws, and *half* by the foot-screws. Now turn the telescope and "upper plate" through 90° over the other foot-screw (or pair of foot-screws) and bring the bubble to the centre of its run with this screw (or pair of screws) *alone*. The vertical axis of rotation will now be nearly vertical. Repeat till perfect. Adjust the levels on the horizontal plates entirely by their own adjusting screws.

4. *Adjustment of Vertical Arc or "Zero of Altitude," i.e.* to make the vernier of the vertical arc read zero when the vertical axis of rotation is truly vertical. If after performing the third adjustment, the vernier of the vertical arc does not read zero, when the telescope bubble is in the centre of its run, then make it do so by adjusting the vernier plate, by means of the small screws by which it is fixed. In lieu of adjusting the vernier plates, the reading of the vernier may be noted and applied as an "*index correction*" to all vertical angles.

Automatic Zero and Right-angle Setting Theodolite.—Messrs. Troughton & Simms, Fleet Street, London, now make a theodolite fitted with special stops to enable the theodolite to be set to zero automatically, and also to enable a right angle to be set off automatically, whereby time can be saved in traversing and setting-out work. Particulars can be obtained from the makers.

TELEMETERS

Telemeters.—The various instruments which have been devised for rapidly determining distances, without chain measurements, may be classed under two principal heads:

(A) Those in which the base is a rod or staff, held at the point whose distance is to be determined.

(B) Those in which the base is measured or fixed at the point of observation. Most military range-finders belong to this category.

Class A may be subdivided into two sub-classes :

(*a*) Those in which the distance is determined by the length on a staff which subtends a constant angle.

(*b*) Those in which the distance is determined by the angle which a constant length on a staff subtends.

We have already studied instruments of this kind in the tacheometer (with or without an anallattic lens) and the subtense instrument. We shall now consider some similar instruments.

The Cleps.—The cleps was invented by Professor Porro, of Milan. It differs considerably from the tacheometer. While the latter has its "horizontal" and "vertical" circles uncovered, as in the ordinary theodolite, in the former they are enclosed in a cubical box. In the cleps, instead of the angle being read by means of verniers, micrometer wires applied to the microscopes are used. The telescope of the tacheometer is "concentric," that of the cleps is "eccentric." The graduation of the horizontal circle in the tacheometer proceeds in the same direction as the hands of a watch, whereas in the cleps it is graduated in the opposite direction. The tacheometer is provided with an ordinary Ramsden eyepiece; the cleps, on the other hand, has a multiple Argo eyepiece. Both instruments have anallattic lenses, and in both the circles may be divided centesimally. Porro used a staff of triangular shape, measuring $2\frac{1}{2}$ inches on each face, and graduated on all three faces. There are different graduations on each of the sides, one with very fine lines for near readings, one in equal blacks and whites, and one divided to meters and decimetres only, for distant readings.

This instrument was used on the Mont Cenis Tunnel Survey. The degree of accuracy obtained was so great that the error never exceeded $\frac{1}{2000}$.

The disadvantages attached to it are its extreme delicacy, and the very careful handling consequently required, the impossibility of testing its adjustments, and the eccentricity of the telescope, which amounts to nearly 3 inches in the larger instruments. The small size of the circle is objectionable, as also are the shadows cast by the windows used for lighting the circles, and which are apt to cause false readings.

The field work with the cleps is practically the same as with a tacheometer.

The Omnimeter.—The omnimeter, by Scholdt, is an instrument which may be used either with a fixed base or with a graduated staff.

This instrument has been much used in India and America, and consists of an ordinary theodolite, with a powerful microscope,

furnished with cross-wires in the focus, attached to the telescope axis, and at right angles to its optical axis By means of this microscope, the graduations of a horizontal scale, fixed on the upper plate of the theodolite, and having a small motion in the direction of the plane on which the telescope revolves, may be read with precision.

The scale is 4 inches long, and is divided into 100 primary divisions, which are again subdivided, making 200 divisions in all.

The micrometer screw has 50 threads per inch, and the head is divided into 100 divisions, and these can be subdivided by the vernier five times. The total subdivision thus becomes 100,000

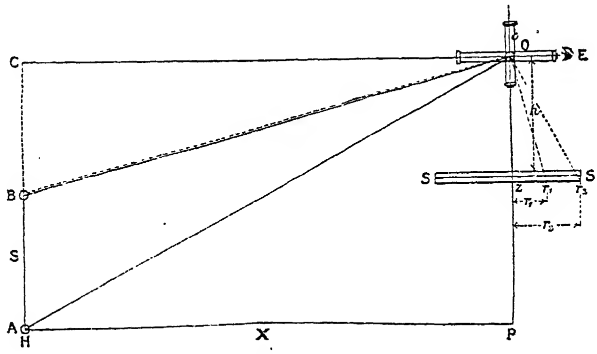

FIG. 9.

in 4 inches, and when the telescope is level, the microscope should read 50,000 on the scale, *i.e.* the central division.

The distance from the centre of the axis of the telescope to the tangent surface of the scale is 6 inches.

The distances may then be obtained by the following geometrical principle. First, when a base of constant length is used.

Let O (Fig. 9) be the centre of the telescope axis. Let OZ be the optical axis of the microscope, which is accurately and rigidly fixed at right angles to the line of collimation of the telescope. Let SS′ be the graduated scale, which must be accurately parallel to the line of collimation of the telescope when the latter is horizontal. The zero Z of the scale must be in the line of collimation of the microscope when it is vertical. Let OC $= x$, the desired horizontal distance.

Let the distance OZ $= h$, a constant quantity. Let $\frac{1}{C} =$ the

ratio of the length of one graduation of the scale, to h the height of the axis above the scale of the telescope.

A base staff S, with marks A and B near to its extremities, is held perpendicularly, at the point whose distance is to be determined. The mark B is now intersected by the horizontal wire of the telescope. The microscope is now focussed to the horizontal scale, and the division of the scale intersected by the microscope wire is read off. To enable the fraction of a scale-division to be read with precision, a micrometer screw is provided, by which the scale can be moved horizontally. The pitch of the screw is such that one turn moves the scale through a space of one division. The head of the screw is divided into one hundred parts. When, therefore, the cross-wire of the microscope falls between two divisions of the scale, the micrometer is turned until the division next below the wire is brought into coincidence with the wire. The unit division is then brought into coincidence with the wire. The whole divisions are read off by the microscope. (See micrometers, Part I.)

Call the first reading r_1; next, let the point A be intersected by the horizontal wire of the telescope, and a second scale reading r_2 obtained.

Then, since the optical axes of the telescopes are always perpendicular to each other, it is evident, by similar triangles, that

$$\frac{BC}{CO} = \frac{r_1}{h} \qquad BC = CO\frac{r_1}{h}$$

$$\frac{AC}{CO} = \frac{r_2}{h} \qquad AC = CO\frac{r_2}{h}$$

$$AC \pm BC = x \times \frac{r_2 \pm r_1}{h}$$

$$\therefore x = \frac{hs}{r_2 \pm r_1}$$

where x is the distance, and s the length of the staff.

The distance can be computed by means of a table of reciprocals, using the plus sign when one angle is an elevation and the other a depression, and the minus sign when both are depressions or both elevations.

The divisions of the scale are so selected as to be some aliquot part of the distance h, from upper plate to collimation axis, usually $\frac{1}{100}$. In any case, r may be written $\frac{hn}{C}$, where $\frac{h}{C}$ is a constant and $n =$ the number of divisions and parts of a division, so that

$$x = \frac{hs}{\frac{h}{C}(n_2 \pm n_1)}$$

$$= \frac{Cs}{n_2 \pm n_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Suppose } C = 100 \cdot 0$$
$$\text{and } s = 10 \cdot 0$$

$$\text{then } x = \frac{1000}{n_2 - n_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Horizontal distances may be read directly, by means of a graduated staff, as follows :—

Direct the telescope to some even division of the graduated staff. Move the scale, by means of the micrometer screw, until the cross-hair of the microscope intersects exactly some one division of the scale.

Without moving the micrometer, move the telescope by means of the tangent screw until the cross-hair of the microscope exactly intersects some other division of the scale, distant n divisions from that first intersected. Read the staff again. The difference of the two staff readings being called s, the distance is obtained as follows :—

$$x = \frac{Cs}{n}$$

$$\text{Suppose that } C = 100$$
$$\text{and } n = 1$$
$$\text{then } x = 100s$$

Differences of level may be obtained by means of the omnimeter. It is evident that the scale-readings are the tangents of the angles of elevations, to base $OZ = h$.

Consequently, the difference of level between O and B is

$$\Delta h = x\frac{r_1}{h} = \frac{x \cdot \frac{hn_1}{C}}{h}$$

$$= \frac{xn_1}{C}$$

where x is given by either of the above rules.

Adjustment of the Omnimeter.—Omnimeters vary considerably in construction, and consequently no general rules can be laid down for their adjustment.

In addition to the ordinary adjustments, common to all theodolites, the following points have to be attended to :—

(1) The "zero" of the "horizontal scale," when the instrument is levelled, should be perpendicularly below the axis of the telescope.

(2) The "optical axis" of the "microscope" should be accurately at right angles to the "optical axis" of the "telescope."

(3) The coefficient of the instrument should be checked.

The adjustment can often be performed as follows:—

Set the instrument up opposite to a vertical wall. and level it carefully. It should be as near to the wall as is consistent with distinct focussing.

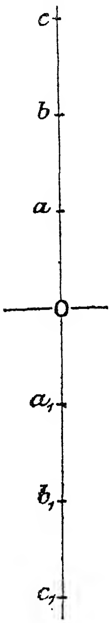

Fig. 10.

Level the telescope and make a fine mark on the wall at O (Fig. 10) to coincide with the horizontal wire of the telescope.

Now make three equidistant marks a, b, and c above O and a_1, b_1, and c_1 below O. Intersect these marks in succession both above and below O, and take the scale reading for each.

If the adjustments are correct the difference of the scale readings will be equal.

If not, then there may be an error either in the zero of the scale or in the perpendicularity of the microscope axis, or both.

If the difference of the scale readings for Oa and Oa_1 differ, but the differences for ab, bc, a_1b_1, and b_1c_1 are equal, then there is an index error of the scale, but if the differences for ab, bc, a_1b_1, b_1c_1, differ from each other, then the microscope axis is not perpendicular to the telescope axis.

Thus, if the difference for ab is *less* than the difference for bc, then the axis of the microscope makes less than a right angle, measured on the side of the object-glass of the telescope.

The coefficient of the instrument is determined by observation to a base placed at a measured distance. It is convenient that the coefficient should be a round number such as 100, so as to facilitate the construction of a reduction table. If on trial the coefficient is not found to be exactly, correct, then the standard table can still be used, by lengthening or shortening the base, as described in connection with the subtense theodolite, Part I.

Comparison with Other Instruments.—It will be seen that the omnimeter is similar in principle to Mr. Wilfred Airy's method with an ordinary theodolite (see Part I.) or to the gradient telemeter level; it differs from each of them in the mechanical arrangements.

The determination of distances by "vertical angles," observed to the extremities of a fixed base, is equally dependent on the optical quality of the telescope with the methods hitherto discussed. Its accuracy depends, principally, upon the exact bisection of the marks on the staff, on the precision with which the "vertical arc" can be read, and on the correctness of the divisions of the limb. With an ordinary theodolite, a single reading of a "vertical angle" can scarcely be depended on to $\pm$ 15″, an angle which subtends 0′·00727 at 100 feet, or an error of $\pm$ 1 in 13760.

When the optical axis is "horizontal" or nearly so, the effect of an "angular error" is practically equal to an error in the length of the "base," equivalent to the distance subtended by the "angular error" at the given distance.

Thus, at 200 feet the resultant error due to an angular error of 15″ is equivalent to an error in the length of the base of 0·0145 of a foot. The base being 10 feet, the proportional error is 0·00145 or 1 part in 687.

The following table shows the *increasing* "proportional error" with the distance measured :—

TABLE OF "MEASUREMENT ERRORS" DUE TO "ANGULAR ERRORS" OF ± 15″.

Angular equivalents.			Measurement errors resulting.
0′·00727	at	100 feet	1 in 1376
0′·01454	,,	200 ,,	1 ,, 687
0′·02181	,,	300 ,,	1 ,, 459
0′·02908	,,	400 ,,	1 ,, 344
0′·03635	,,	500 ,,	1 ,, 275
0′·07271	,,	1000 ,,	1 ,, 137

It is therefore clear that this method, with an ordinary instrument, is not capable of high accuracy, except at short distances.

In the above table a 10-foot staff base is assumed. With other bases the figures will be different, of course, but the principle will be the same, so long as the base is constant.

Both with the "omnimeter" and the "subtense instrument" any given error in the measurement of the angle (or rather of its tangent in the case of the first-named instrument, and its chord in the second) produces a rate of error in the result which increases in proportion to the distance measured, just as in the case of measurement by vertical angles. Thus, if it were found that at 1000 feet an error of ± 1 per thousand is probable, the probable error at 2000 feet would be ± 2 per thousand, and so on.

The "omnimeter" measures the "tangents of angles" mechanically with high precision. The optical quality of the telescope has little influence on the results. Theoretically, the "omnimeter" should give a very high degree of accuracy, superior in fact to the other instruments described. In practice, this has not been found to be the case, owing to mechanical difficulties connected with the micrometer screw, also in that the distance between the microscope and the scale is variable, whilst the scale itself is not sufficiently finely and accurately graduated. To focus the horizontal scale at different points of its length, the object glass, or eyepiece of the microscope, or both, must move through a considerable space. A

slight want of truth in the draw-tube of the microscope will produce a material error in the measured tangent. In this instrument, also, the error-fraction due to any given error in measuring the tangent increases in direct proportion to the distance measured. The "omnimeter" has this merit, that the reduction of the observation is extremely simple. By using a graduated staff and a fixed difference of tangent, horizontal distances may be read directly without any reduction. If all mechanical difficulties were overcome, the "omnimeter" would be the most rapid and accurate of all the "telemetric instruments" described.

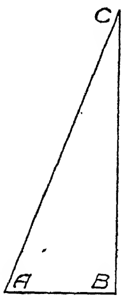

FIG. 11.

Range-finders.—The term *range-finder* is a general name applied to instruments for finding quickly the distance of an inaccessible object. In principle the great majority work by an indirect means of measuring the angle BAC (Fig. 11), at the base of a right-angled triangle, where BC is the distance required. The base AB may be a fixed length on the instrument itself, or a variable measured base. It is clear that BC = AB tan BAC, and the instrument is often graduated to give tan BAC, or BC directly if AB is fixed.

They are hardly surveying instruments as an engineer would understand the term, but as they are sometimes used in exploratory work, etc., a few examples are here given, illustrating different methods of measuring the angle.

The Barr and Stroud.—The Barr and Stroud range-finder is an example of those with a fixed base. In this instrument the rays of light from the object are reflected from mirrors M_1, M_2 (Fig. 12), through the object glasses O_1, O_2, which form images viewed through the eyepiece at E by reflection from the mirrors N.

The mirror M_1 is fixed; and as the mirror M_2 is turned to bisect objects at different distances, the image formed by the lens O_2 is displaced and ceases to coincide with that formed by the lens O_1. Coincidence is re-established by shifting the wedge W from left to right, or *vice versâ*, and the amount of shift necessary gives a measure of the angle through which the ray is turned, and therefore of the distance.

For a full description see *Proceedings of the Institution of Mechanical Engineers* for 1896.

The degree of accuracy aimed at in designing the instrument was 1 per cent. at 3000 yards range, which is almost exactly equivalent to detecting a movement of one second of arc.

The Reeves Distance-finder.—This instrument is made by

Messrs. C. F. Casella & Co., Whitehall, and is shown in Fig. 13. The following short description is abridged from the illustrative pamphlet issued by the makers.

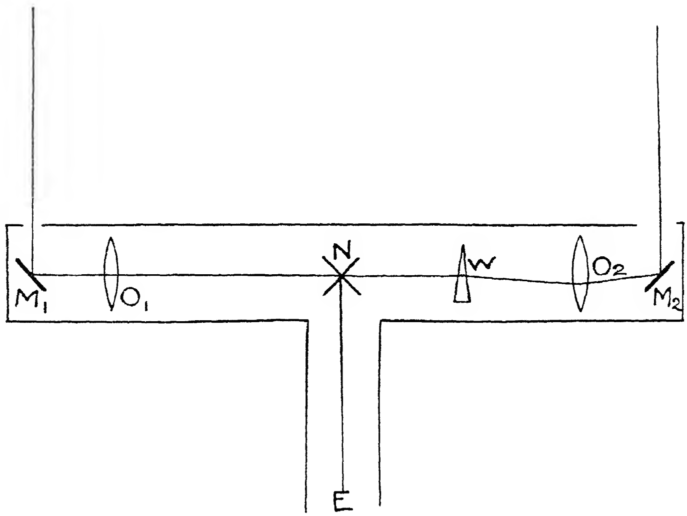

Fig. 12.

The instrument consists of a bar, which may be of invar carrying two telescopes set at right angles to it.

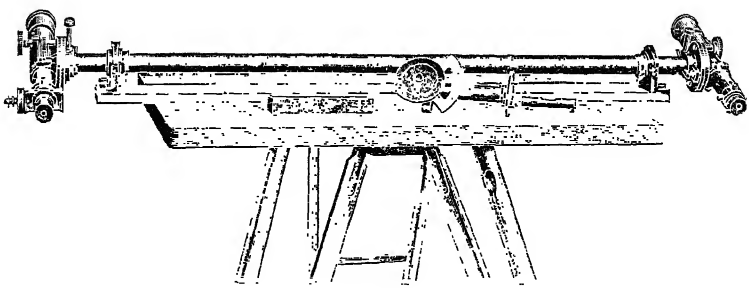

Fig. 13.

The bar can be revolved horizontally and levelled, and can also be taken off and turned upside down. Each telescope can also be reversed, so that errors of adjustment can be eliminated.

One telescope has only fixed horizontal and vertical hairs. This one is permanently perpendicular to the bar, and is first set on the object by turning the whole instrument. The other telescope has, in addition, a movable vertical hair worked by a micrometer, and, after the first telescope is set, this hair is also brought to coincide with the object by the micrometer. The reading of the latter clearly affords a measure of the distance in terms of the length of bar. The latter may be $2\frac{1}{2}$ feet or 5 feet. For further particulars see the makers' pamphlet.

The Hymans Pocket Range-finders.—In this instrument, which is made by Mr. Chas. Hymans, 7, St. Andrews Street, Cambridge,

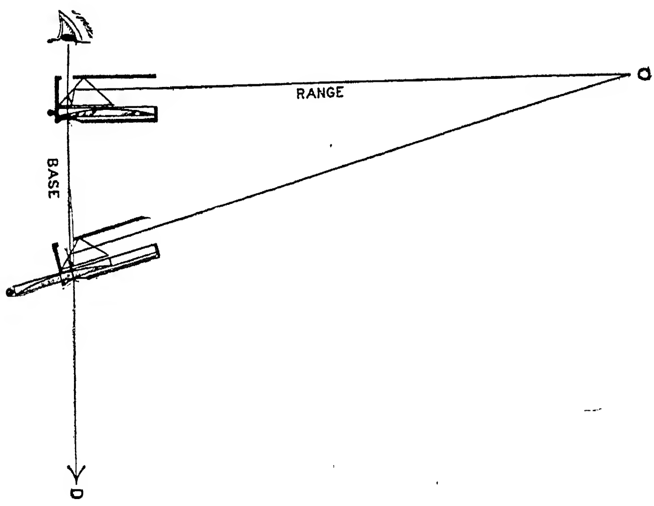

Fig. 14.

coincidence is established at each end of a measured base, between any convenient fixed object nearly in the line of the base and the object whose range is required, which is the right or left of the base. Coincidence having been made with a suitable reference object at one end of the base, it is re-established at the other end by sliding out a flat cylindrical lens, as illustrated in Fig. 14. Full particulars may be obtained from the maker.

Some tests by the writer with this instrument gave better results than those claimed by the maker, but much depends upon the accuracy with which the length of the base is known.

The Steward Surveying Telemeter. (Fig. 15)

It is claimed for this instrument that by its use distances may be measured with considerable accuracy, and that surveys may be made with it without any other optical instrument being employed. The makers quote nineteen observations where the average length per observation was 1728 feet, the longest measured 2786 feet, and the shortest 649 feet. The average plus error was 16·6 feet, the average minus error 17·4 feet, and the average error was 17·1 feet, or 1 per cent. nearly.

Description of Instrument.—The instrument consists of a cylindrical case containing two mirrors, both of which are adjustable. The instrument is designed to measure the two angles at the extremities of a base opposed to the object observed.

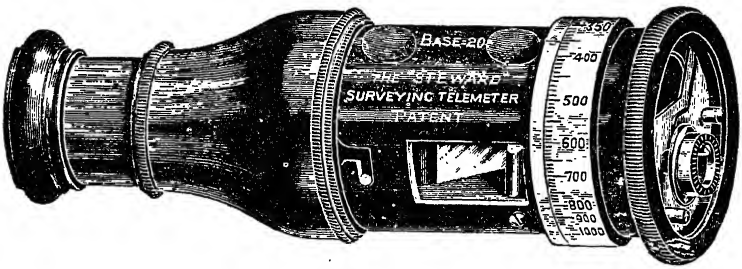

Fig. 15.

By rotating a collar at the end of the tube the index mirror is moved in azimuth, and the angular displacement is measured by reference to the distances graduated on the exterior of the collar.

The angle of double reflection can be varied several degrees on either side of the right angle, facilitating the determination of the direction of the base.

The range is read directly on the graduated scale engraved on the collar, in terms of units of the base.

The base, if short, may be measured with a tape, and should be approximately at right angles to the observation line.

Longer bases may be measured with the instrument itself from an original tape measurement.

Precise Levels

"Precise" levels are used in levelling operations for detecting movements of the earth's crust, and in other cases where minute accuracy is desirable.

They differ from the levels previously described chiefly in the facts that they are made of invar, that the telescope is more powerful and the bubble more sensitive, and that they are fitted with stadia lines, so that, by reading these the distance of the staff can be found nearly enough for the equalization of back- and fore-sights.

Arrangements of prisms or mirrors are provided so that the bubble can be seen at the same time as the reading is being taken, and in some cases there is a micrometer for estimating fractions of a staff division. In the latest Ordnance Survey practice this enables a reading to be estimated to the fourth decimal of a foot.

The spirit level is sometimes separate from the telescope, so that it can be reversed, as already described in Part I., or sometimes it is sunk to the level of the telescope axis in order to minimize temperature errors.

The telescope now most frequently used is of the Zeiss pattern (see Part I.), with a magnification of about 30 diameters.

The staff is less flexible than an ordinary levelling staff, and usually differently graduated. In the latest Ordnance Survey pattern the graduations are inscribed on a strip of invar let into a wooden staff, each division being 0·02 foot and the separating lines $\frac{1}{40}$ inch broad. The lines are black, the spaces white. The whole rests on an invar base. Ordinary wooden staves require correction for temperature, moisture, etc. Special supports (usually driven into the ground) are provided for the staff.

The bench-marks fixed for future reference in connection with such levelling must, of course, be very different (alike from the points of view of accuracy and permanence) from the rough arrow-heads to which we are accustomed.

In the Ordnance Survey practice the length of a sight is limited to 150 feet, and the distances for back- and fore-sights must not differ by as much as 3 feet. Two staves are used, so that the readings can be taken with the least possible delay between back- and fore-sight.

All lines should be levelled in each direction.

The adjustments of the instrument and of the staff levels are tested weekly.

For farther information on these matters the reader is referred to a paper on "Precise Levelling" by Major Henrici in the *Proceedings of the Institution of Civil Engineers* for 1919–20.

CHAPTER II

SOME MATHEMATICAL FORMULÆ

It has been assumed, throughout this work, that the student is familiar with the ordinary rules of algebra, and with the elements of plane geometry and trigonometry. These supply all the mathematical knowledge required for the survey of a small area by such simple methods as we have hitherto discussed.

Such simple rules are, however, insufficient for even the elementary study of the operations involved in the survey of a large area, or for what are called *Geodetic Surveys,* of which one object is the determination of the exact form and dimensions of the earth as a whole.

It is proposed here to write down a few of the most commonly useful formulæ. For proofs of these the student is referred to works on mathematics. They are merely collected here as a matter of convenience.

Spherical Geometry.—If any sphere be intersected by a plane which passes through the centre of the sphere, the circle in which that plane meets the sphere is called a "great circle." If the plane of the circle does not pass through the centre of the sphere, the circle is called a "small circle." Thus, in Fig. 16, ABCD, AKCR, and KBRD are great circles, but KLMN is a small circle.

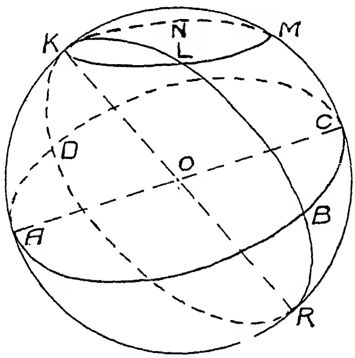

Fig. 16.

"Straight Line" defined.—If the surface of the earth were truly spherical, then any "straight line" laid out upon that surface by any ordinary process of surveying, would be a portion of a great circle. If it could be constantly prolonged without error, it would eventually return to its starting point, having on

the way passed through the opposite extremity of the diameter through that point. The whole length of the line would lie in one plane through the centre of the sphere, and this plane would contain the verticals at all points on the line; also if any two points on the line were taken, the distance between them as measured along the straight line so laid out would be less than the distance between the same points as measured along any other line on the surface of the sphere; and if a theodolite were set up at any point on the line, and directed in azimuth to one end thereof, then the vertical plane described by the theodolite would pass also through the other end of the line.

The fact that all these properties are united in the same line justifies us in deciding that by the *line AB*, in surveying, we shall understand the arc of the great circle between A and B, assuming the earth spherical.

On Spheroidal Surface.—Actually, however, the polar diameter of the earth is shorter than the equatorial, and the true shape of the earth is not spherical. It therefore becomes impossible to find any one line which shall combine all the above properties, and it is not easy to decide exactly what is to be understood by the line AB.

Four separate lines may be considered—

1. The line in which the spheroidal surface is intersected by a plane which is vertical at A and passes through B.

This is the line which would be laid out by a theodolite set up at A and directed to B.

2. The line in which the surface is cut by a plane which is vertical at B and passes through A.

These two lines are both plane curves, but they do not, in general, coincide with one another, and if a theodolite be set up at any point in either and directed, say, to A, then the vertical plane described by the instrument will not pass through B.

3. The "curve of alignment." The property chosen for fixing this line is that if a theodolite be set up at any point in it and directed say to A, then the vertical plane described by the instrument shall pass through B also.

It is evident that this line is tangential to each of the above at the ends A and B respectively. But it does not lie in one plane, and none of the three gives the *shortest distance* between A and B.

4. The *geodetic line*. The property whereby this is chosen is that the distance between A and B, as measured along it, shall have the least value.

The consideration of these lines is beyond the scope of this work. In any practical case there is very little difference between their

lengths, and for most purposes it is sufficient to treat any triangle as lying on a sphere whose radius is equal to the mean radius of the earth's curvature at the centre of the triangle.

For the further study of these lines and of triangles on the spheroid, the reader is referred to the treatise on Geodesy by Col. A. R. Clarke (Clarendon Press series).

Spherical Triangle.—If any three great circles on a sphere intersect one another, the intercepted arcs form a *spherical triangle*.

Thus, in Fig. 16, RCB and KAB are spherical triangles, but KLM is *not*, because the arc LM is not part of a great circle.

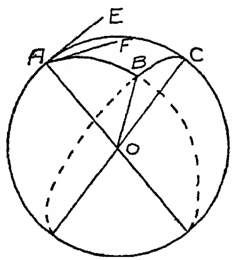

Fig. 17.

Let ACB (Fig. 17) be any spherical triangle, O the centre of the sphere.

Then each *side* of the triangle is measured by the *angle* it subtends at the centre O of the sphere, *not* by its actual length, as in plane trigonometry.

Thus the side CB is measured by the angle COB, and we would say that side is 30°, or whatever the angle might be. Clearly to find the actual length we must know the radius of the sphere, then—

Length of side CB = radius of sphere × angle COB in radians
= ,, ,, × side CB of triangle

the latter being expressed here in radians.

The *angles* of the triangle are measured by the *angles between the planes* in which the sides lie.

Thus the angle A of the triangle ACB is the angle between the planes AOC and AOB.

If we draw AE, AF, tangents at A to AC and AB, the angle between these tangents will give the angle between the planes, and hence the angle of the triangle.

Formulæ of Solution.—In writing down the formulæ for solving spherical triangles, we shall adopt the usual notation common to plane and spherical trigonometry. That is, the side opposite the angle A is called a, and so on. The *half-sum* of the sides is called s, and is, of course, an angle.

Then the formulæ of solution most commonly required are—

$$\cos a = \sin b \sin c \cos A + \cos b \cos c \quad . \quad . \quad . \quad (1)$$

This enables us to find the third side when two sides and the included angle are given, or to find an angle given the three sides.

These give the remaining sides when two angles and the side between them are known.

We cannot find the third angle as in plane trigonometry, by subtracting the sum of the known angles from 180°, in consequence of the spherical excess (p. 33). Otherwise we could proceed by No. (3).

Right-angled Triangles.—Right-angled triangles can be solved by putting A, B, or C, as the case may be, equal to 90° in these formulæ.

Six different formulæ result, any one of which may be required to solve any given case.

These formulæ have, however, been combined in *two* simple rules (which the student should carefully learn) by making certain conventional agreements.

These are—

(*a*) We agree to ignore the right angle, so that the triangle is regarded as having only *five* parts, arising from *three sides* and *two angles*.

(*b*) The sides *adjacent* to the right angle shall have their proper values, but the remaining three parts shall be found by taking the *complements* of the hypotenuse (that is, the side opposite the right angle) and of the two angles.

Thus if ABC, Fig. 17, be the triangle, and B be the right angle, the five parts are: (1) the side BA; (2) 90° − A; (3) 90° − AC; (4) 90° − C; (5) CB.

We may write these down in five successive sectors of a circle as shown in Fig. 18, taking care to put them in proper order as they come, one after the other, in passing round the triangle (in either direction) from the right angle.

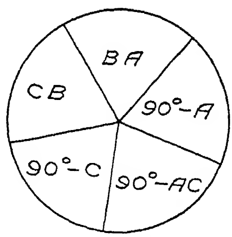

Fig. 18.

(*c*) Lastly, we agree that if we start from any one of these parts so written down as a *middle* part, the two which are adjacent to it shall be called *adjacent* parts, and the remaining two shall be called *opposite* parts.

The two rules are then as follows :—

sine of middle part = product of cosines of opposite parts

and *sine of middle part = product of tangents of adjacent parts*

As a help in remembering these formulæ, note that *cosines* and *opposite* both have *o* for their first vowel, and they go together; likewise, *tangents* and *adjacent* have *a*.

These give the remaining sides when two angles and the side between them are known.

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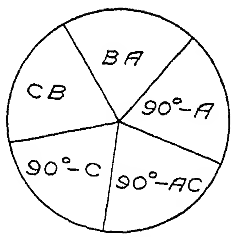

Fig. 18.

(*c*) Lastly, we agree that if we start from any one of these parts so written down as a *middle* part, the two which are adjacent to it shall be called *adjacent* parts, and the remaining two shall be called *opposite* parts.

The two rules are then as follows :—

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and *sine of middle part = product of tangents of adjacent parts*

As a help in remembering these formulæ, note that *cosines* and *opposite* both have *o* for their first vowel, and they go together; likewise, *tangents* and *adjacent* have *a*.

To use the formulæ, we must be given any two parts, and we require to find any third part, so that we have to consider *three* parts in any given case.

All these parts being written down as in Fig. 18, it will be found that we can take one (and only one) of the three parts as *middle*, so that the other two are *both* opposite, or *both* adjacent. We then apply the proper formula.

Thus suppose $B = 90°$, given AC and AB, required to find the angle C.

From Fig. 18 it is clear that if we take $90° - C$ as middle part, $90° - AC$ is adjacent and BA is opposite. But if we take BA as middle, $90° - C$ and $90° - AC$ are both opposite, and we use the *first* of the above formulæ, whence

$$\begin{aligned} \sin BA &= \cos . (90° - AC) \cos (90° - C) \\ &= \sin AC . \sin C \\ \text{Hence } \sin C &= \sin BA . \operatorname{cosec} AC \end{aligned}$$

Similarly, given BA and AC, required the angle A.

Here the figure shows that we must take $90° - A$ as middle, when BA and $90° - AC$ are both adjacent.

Hence by the second formula—

$$\begin{aligned} \sin (90° - A) &= \tan BA . \tan (90° - AC) \\ \text{or } \cos A &= \tan BA . \cot AC \end{aligned}$$

These formulæ are much less difficult to learn and remember than one would suppose, and the student should deduce from them the usual formulæ, which are here written down on the assumption that $C = 90°$ (so that Fig. 18 must be redrawn accordingly):

$$\begin{aligned} \sin a &= \sin c \sin A \\ \tan a &= \sin b \tan A \\ \cos c &= \cos a \cos b \\ \cot A &= \cos c \tan B \\ \cos A &= \cos a \sin B \\ \tan a &= \tan c \cos B \\ \text{Similarly, } \tan b &= \tan c \cos A \end{aligned}$$

and so on for the other formulæ in which we write b for a and B for A, or *vice versâ*.

Examples.—As examples in the use of these formulæ we shall take the following, in which the earth is assumed to be spherical.

1. At a point A in latitude 50° N. a line is set out due east at A, and carried straight on for 60 nautical miles to B. Given that one nautical mile subtends one minute at the earth's centre, find the azimuth of the line AB at B—that is, the angle which this line makes with the meridian at B—and the latitude of B.

Here, if P (Fig. 19) be the north pole, and E on the equator—

$$\begin{aligned} EOA &= \text{latitude of } A = 50^\circ \\ \text{and } POA &= 90^\circ - EOA = 40^\circ \end{aligned}$$

But this is the measure of the side AP.

$$\begin{aligned} \text{Hence } AP &= 40^\circ \\ AB &= 1^\circ = 60 \text{ nautical miles} \\ \text{and } PAB &= 90^\circ, \text{ as } AB \text{ is due east at } A \end{aligned}$$

The parts are written down as in Fig. 20.

The first problem is, given AP and AB, to find the angle ABP, which gives the azimuth of BA at B.

An examination of Fig. 20 shows that if AB be taken as the middle part, AP and 90° – B are *both adjacent.*

Hence the formula is—

$$\begin{aligned} \sin AB &= \tan AP \,.\, \tan (90^\circ - B) \\ \text{hence } \cot B &= \sin AB \,.\, \cot AP = \sin 1^\circ \,.\, \cot 40^\circ \\ \therefore B &= 88^\circ\ 48'\ 30'' \text{ nearly} \end{aligned}$$

that is, BA is 88° 48′ 30″ *west* of north at B. The azimuth of AB at B differs from this by 180°, hence AB is 91° 11′ 30″ *east* of north at B.

The second problem is, given AP and AB, to find PB. (Note PB $= 90^c -$ latitude of B.)

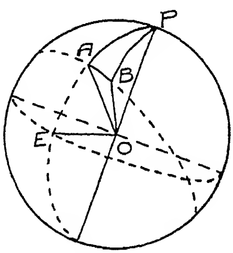


FIG. 19.

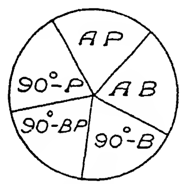


FIG. 20.

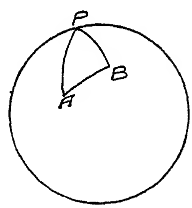


FIG. 21.

From Fig. 20 we see that we must take 90° – PB as middle part, when AP and AB are both *opposite.*

$$\begin{aligned} \text{Hence } \sin (90^\circ - PB) &= \cos AP \,.\, \cos AB \\ \therefore \cos PB &= \cos 40^\circ \cos 1^\circ \\ \therefore PB &= 40^\circ\ 0'\ 37'' \\ \text{hence latitude of } B &= 49^\circ\ 59'\ 23'' \text{ N.} \end{aligned}$$

Thus a straight line which is due east at A does not continue to run due east.

(2) Two points, A and B, whose latitudes are 50° and 51° N. respectively, are to be joined by a straight line. The difference of longitude is 2°. Find the length of the line AB (in nautical miles), and the azimuth of the line at A and at B (Fig. 21).

$$\begin{aligned} \text{Here } PA = 90^\circ - 50^\circ = 40^\circ; \quad PB = 39^\circ \\ \text{and } BPA = 2^\circ \end{aligned}$$

(*a*) To find AB.

$$\cos AB = \cos PA \,.\, \cos PB + \sin PA \,.\, \sin PB \,.\, \cos P \ldots \text{(p. 33)}$$
$$= \cos 40^\circ \,.\, \cos 39^\circ + \sin 40^\circ \,.\, \sin 39^\circ \,.\, \cos 2^\circ$$

No.	Log.	Antilog.
Cos 40° Cos 39°	9·8842540 9·8905026	
	9·7747566	0·595328
Sin 40° Sin 39° Cos 2°	9·8080675 9·7988718 9·9997354	
	9·6066747	0·404273
	cos AB =	0·999601

This gives the result AB = 1° 37′ 5″ = $97\frac{1}{12}$ nautical miles.
Azimuth of AB at A = angle PAB.

$$\text{Now } \frac{\sin PAB}{\sin BPA} = \frac{\sin PB}{\sin AB}$$

$$\therefore \sin PAB = \frac{\sin 39^\circ}{\sin 1^\circ\ 37'\ 5''} \times \sin 2^\circ$$

$\therefore$ PAB = 51° 3′ 42″, east of north if the points be situated as shown in the figure.

$$\text{Similarly, } \sin ABP = \frac{\sin 40^\circ}{\sin 1^\circ\ 37'\ 5''} \times \sin 2^\circ$$

$\therefore$ ABP = 127° 23′ 42″ = azimuth of BA, west of north

These angles give the directions in which we must set out at A and B, when we should meet after working a total distance of $97\frac{1}{12}$ nautical miles.

(Note.—The student who works through this exercise will find that the tables give the angle APB as 57° 36′ 18″. It is to be remembered, however, that sin A = sin (180° − A). Thus an angle found from its *sine* may have either the value given in the tables or the supplement thereof. In this case *one* of the angles PAB or ABP must be greater than 90° to make the sum of the three angles not less than 180°; and as AP is greater than BP, the angle ABP must be greater than PAB, for in spherical trigonometry, as in plane, the greater side of the triangle must have the greater angle opposite to it. Hence we take the supplement of the above angle.)

This case would be better solved by the logarithmic formulæ (5) on p. 34.

Spherical Excess.—The sum of the three angles of a spherical triangle can be shown to *exceed* 180° by an amount called the "spherical excess," and given by the formula

$$360^\circ \times \frac{\text{area of the triangle}}{\text{area of the hemisphere}}$$

Thus in Fig. 16, p. 31, if KBC be a spherical triangle in which each side is a quadrant of a circle, each angle will be 90°. Hence the sum = 270°, and spherical excess = 90°.

It is easy to show that the area of such a triangle is *one quarter* that of the hemisphere, which verifies the formula.

As any figure on the sphere bounded by arcs of great circles can be divided into triangles, it follows that the sum of the angles of such a figure is *greater* than $2n \times 90° - 360°$ (which would be the sum for a plane figure, if n be the number of sides) by an amount given by the formula

$$360° \times \frac{\text{area of the figure}}{\text{area of the hemisphere}}$$

Hence, strictly, the angles of an extensive traverse should not add up to twice as many right angles as the figure has sides, plus or minus 360°, as we have assumed in Part I.

For an area of 100 square miles, however, the spherical excess is only about 1·3 seconds, which is negligible in ordinary traverse work, and in any case the figure must be reduced to a plane figure to be plotted on paper.

In calculating the spherical excess it is sufficiently near to find the area of the figure as if it were a plane surface.

That is, if E = spherical excess, r = mean radius of earth's curvature at the place, a, b = two sides of triangle C = included angle, then

$$\mathrm{E} = \frac{ab \sin \mathrm{C}}{4\pi r^2} \times 360°$$

$$\text{Now } 360° = 2\pi \text{ radians}$$

$$\text{hence, in radians, } \mathrm{E} = \frac{ab \sin \mathrm{C}}{2r^2}$$

To bring this to seconds divide by sin 1″,

$$\therefore \mathrm{E} = \frac{ab \sin \mathrm{C}}{2r^2 \sin 1''} \text{ secs.}$$

Legendre's Theorem.—In calculating a large number of small triangles, as in an ordinary triangulation, if the sides be computed by spherical trigonometry, it is necessary to convert each one from an angle to an actual length of arc. This involves considerable labour; hence surveyors frequently make use of the rule known as Legendre's theorem, namely, that if each of the angles of a spherical triangle be reduced by *one-third* of the spherical excess proper to the triangle, then, given any one side of the triangle, the remaining sides may be calculated as if the triangle were a

plane triangle having these reduced angles; and the lengths obtained will be the same as if the triangle were computed as spherical.

It is presupposed that the area of the triangle is small compared with that of the hemisphere, but this would be true of any triangle on the earth, whose angles could be observed as part of a triangulation.

Indeed, in triangulation surveys of the kind treated in this book, the spherical excess never exceeds one second, and all triangles are treated as plane, as described later.

Accumulated Error.—The law of the probable accumulation of error has already been referred to under Traverse Surveying in Part I.

If n measurements be made, each subject to an accidental error e, the probable accumulated error is $e \times \sqrt{n}$.

This is one reason why it is more accurate to find the difference in level between two points, at a long distance from one another, by a number of short readings with a level than in one long reading by a vertical angle and known distance.

Assuming that all instrumental and other systematic errors are eliminated in each case, and taking it for granted that the distance is exactly known, we will suppose that the probable accidental error is the same, at the same distance, for the theodolite and level. Let D be the whole distance, and suppose there are n equal readings with the level, so that the distance for each is $\frac{D}{n}$. Let the error for this distance be e. Then the accumulated error by the level will "probably" be $\sqrt{n} \times e$. But it is clear that, for any error in the vertical angle, the error in level is proportional to the distance, and hence the whole error of this method will be $e \times D \div \frac{D}{n}$, or $n \times e$.

Adjustment of Errors.—One of the most important operations in accurate work is the mathematical treatment of the directly observed quantities, so as to eliminate, as far as possible, the unavoidable small errors of observation which occur even in the best work, and to test the reliability of the results.

A complete explanation of such treatment is far beyond the scope of this work, and the reader who wishes to pursue this subject farther is referred to the treatise on "The Adjustment of Observations," by T. W. Wright, published by D. van Norstrand, New York; Clarke's "Geodesy," published at the Clarendon Press, Oxford, and other works.

A few rules for simple cases will, however, be given here.

Accidental Errors.—Let it, first, be quite clear that it is assumed that all errors from known causes have been eliminated by the method of taking the observations (at least so far as this is possible) or allowed for by subsequent correction. That, for instance, if the observations are those of angles, the centering is exact, the signals vertical, the readings taken by several *reiterations* (Part I.) with different directions of rotation, etc.; that if a length has been measured, corrections have been applied for temperature, slope, etc.; that, if the observations require it, the work has been done by several observers to minimize the effects of personal bias, and so on.

As Mr. Wright says, "In trying to avoid systematic error the observer will, as he gains in experience, take precautions which would at first seem to be almost childish. Good work can only be done at the cost of eternal vigilance."

Any error due to neglecting such precautions as above described would be a *recurring* or *systematic* error. If we took n measures of the observed quantity, under the same conditions, the accumulated error would be $n \times$ the error in one observation, and the mean of the n observations would be no better, so far as that particular error was concerned, than one single observation.

The errors which remain after these are eliminated, and to which no definite cause can be assigned, are called *accidental errors.*

The sign of such an error in any given observation may be either *plus* or *minus*, and it is to these errors alone that we can apply the results of the theory of probability.

Sometimes the term *error* is restricted to what we have here called accidental errors. It is convenient, however, to use the word "error" in the wider sense here adopted, and this has been done throughout this book.

The problem is, then, given a series of observations all subject to accidental errors, to find that exact value (or those exact values, if more than one quantity is to be found) which are in most probable accordance with the observations as a whole, and to form an estimate of the accuracy of the observations themselves.

Method of Least Squares.—The method almost invariably adopted is that of "least squares."

Referring to Fig. 22, suppose that a set of values of two quantities x and y (connected together by some law) have been observed, and that their values are shown by the black dots in the figure.

Now suppose that AB is a continuous line which has been

chosen to agree nearly with these points, and to satisfy the known law.

Let any point, such as No. 1, which is *below* this line, be said to have a *positive* error (following out the principle laid down on p. 17, with regard to clock error), then a point such as No. 3, which is *above* the line, has a *negative* error, and the amount of error in each case is denoted by $1a$ or $3b$.

Then mere common sense would suggest that the line should be chosen so that the sum of the positive errors should be equal to

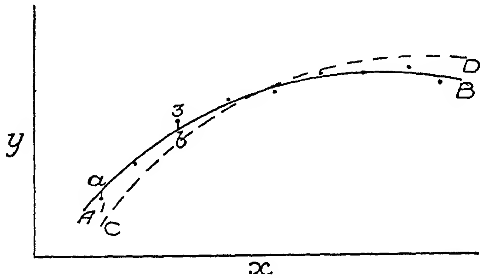


FIG. 22.

the sum of the negative ones, thus giving a fair average value as it would appear at first sight.

This condition is, indeed, generally true for the most plausible line, but it is not actually *sufficient* to fix the line.

Thus it might be quite possible to find a line like CD which would satisfy this condition, and yet is clearly not the best line.

Suppose, however, we *square* each of the small errors. The result will be positive whatever be the sign of the error itself, and it is clear that if we arrange that *the sum of the squares of these errors shall have its minimum value*, we obtain a line which very closely follows the observed points.

This is the principle of least squares.

Directly Observed Quantities.—Now suppose the observations with which we are concerned are a direct set of measurements of one quantity only, say a single angle, and suppose it is measured n times, the results being $M_1, M_2, \ldots M_n$.

Let the errors in these be $v_1, v_2 \ldots v_n$, and let M_0 be the true value of the quantity.

Then our equations are

$$M_0 = M_1 + v_1$$
$$M_0 = M_2 + v_2$$

and so on.

We shall write $[vv]$ to mean $v_1^2 + v_2^2 + \ldots$; $[MM]$ for $M_1^2 + M_2^2 + \ldots$; $[M]$ for $M_1 + M_2 + \ldots$, and so on,

Then the best value we can choose for M_0 is, according to the principle of least squares, that which makes $[vv]$ a minimum.

$$\begin{aligned} \text{Now} \qquad v_1 &= M_0 - M_1 \\ \therefore v_1^2 &= M_0^2 + M_1^2 - 2M_0M_1 \\ v_2^2 &= M_0^2 + M_2^2 - 2M_0M_2 \end{aligned}$$

and so on to n equations.

$$\begin{aligned} \text{Hence} \qquad [vv] &= n \times M_0^2 + [MM] - 2M_0[M] \\ \therefore \frac{d[vv]}{dM_0} &= 2n \times M_0 - 2[M] \end{aligned}$$

Equating to zero to make $[vv]$ a minimum, we have $M_0 = \frac{[M]}{n}$.

Hence the most likely value of M_0 is found by adding the measured values together and dividing by the number.

In other words, it is the simple *arithmetic mean*, a result we should naturally have expected.

Measures of Precision.—The value so found, however, is of course not the *true* value in general. More or fewer measures would have given slightly different results.

It is desirable to have some kind of recognised test of precision to be applied to the observations.

The criteria most frequently adopted are called the *mean square error* and the *probable error* respectively.

Mean Square Error.—Suppose the true value of the quantity considered be known; and let $\Delta_1, \Delta_2, \ldots$ be the true values of the errors in M_1, M_2, etc., obtained from it.

Then putting $[\Delta\Delta]$ for $\Delta_1^2 + \Delta_2^2 + \ldots$ and μ for the mean square error of one observation,

$$\mu = \sqrt{\frac{[\Delta\Delta]}{n}}$$

But as Δ_1, Δ_2, etc., are unknown, we cannot use this formulæ. It is given to explain the name, as it is clear that μ^2 gives the *mean* value of Δ^2. But if $v_1, v_2, \ldots$ be the residual errors (that is the differences, between the arithmetic mean and the separate measures M_1, M_2, etc.), it can be shown that on the rules of probability the formulæ can be reduced to

$$\mu = \pm \sqrt{\frac{[vv]}{n-1}}$$

The corresponding error in the arithmetic mean M_0 according to the theory of probability is then $\mu_0 = \pm\sqrt{\frac{[vv]}{n(n-1)}}$

The residuals v_1, v_2, etc., can be found by subtracting each measure from the mean, then $[vv]$ is the sum of the squares of these, and μ_0 can thus be calculated. It gives one measure of the precision.

Probable Error.—If all the values of Δ were written down in ascending or descending order of arithmetical value (without respect to sign), then that error which would occupy the middle place in this series (assumed infinitely long) is called the *probable error.*

In any finite series of observations its most likely value is found from the mean square error, μ or μ_0, by multiplying by 0·6745.

$$\therefore \text{ probable error of one measure} = \pm\, 0{\cdot}6745\sqrt{\frac{[vv]}{n-1}}$$

$$\text{,, ,, ,, mean} = \pm\, 0{\cdot}6745\sqrt{\frac{[vv]}{n(n-1)}}$$

The value for one measure so found should, in most reasonably good sets of observations be approximately equal to the value of that residual, v_m, which occupies the middle place in a series where all values of v are shown in ascending or descending order of arithmetical value.

Mr. Wright points out that the name *probable error* is a poor one, and that a better name would be the *median error* as proposed by Cournot, or *critical error* as proposed by De Morgan. The present writer thinks that the best name would be the *probable median error.*

At all events it is called the probable error, and is generally adopted in some countries as the criterion of precision, while in others the mean square error is chiefly used.

Example.—For example, suppose 12 measures of a single angle, read by different observers under different conditions, and otherwise taken so as to eliminate systematic error as far as possible, give the results shown in the table. (*Note.*—The degrees are supposed to be constant, and are not shown.) Then the value to be adopted is the arithmetic mean, or 27′ 1·92″.

No.	Angle. ′ ″
1	27 2
2	27 6
3	27 7
4	26 58
5	27 3
6	26 57
7	27 0
8	27 5
9	26 59
10	27 0
11	27 4
12	27 2
Mean	27 1·92

To find the residuals we subtract each measure from this. The results are as follows:—

No.	v.	v^2.
1	− 0·08	0·0064
2	− 4·08	16·6464
3	− 5·08	25·8064
4	+ 3·92	15·3664
5	− 1·08	1·1664
6	+ 4·92	24·2064
7	+ 1·92	3·6864
8	− 3·08	9·4864
9	+ 2·92	8·5264
10	+ 1·92	3·6864
11	− 2·08	4·3264
12	− 0·08	0·0064
		112·9168

Hence μ = mean square error for one observation

$$= \sqrt{\frac{112{\cdot}92}{11}} = \pm 3{\cdot}2''$$

μ_0 = mean square error of arithmetic mean

$$= \sqrt{\frac{112{\cdot}92}{12 \times 11}} = \pm 0{\cdot}92''$$

The probable errors are found by multiplying these by 0·6745.

$\therefore$ probable error of one observation $= \pm 2{\cdot}16''$
and probable error of mean $= \pm 0{\cdot}62''$

If the values of v be written down in ascending order of arithmetical magnitude, the two which occupy the middle places are 2·92 and 2·08; it will be seen that the probable error of one observation lies between these.

The sum of the positive values of v is 15·60; that of the negative values is $-15{\cdot}56$. These should be equal. The discrepancy here is due to the fact that the mean angle is not exactly $27'\ 1{\cdot}92''$. If this be multiplied by 12, the result is $324'\ 23{\cdot}04''$, whereas the sum of the measured angles is $324'\ 23''$. Hence the discrepancy between the positive and negative values of v should be $0{\cdot}04''$, and the negative values should be the smaller as the sum of the measured angles is the smaller of the above.

Indirectly Observed Quantities.—Suppose that values of a quantity Q are observed, and that Q is connected with other quantities K, L, M by the known law $Q = aK + bL + cM$.

Here a, b, c are constants whose values are to be found in order that the law may be known not only in *form* but with numerical coefficients.

K, L, M are other quantities which are either observed at the same time as Q or are known independently; or they may be known functions of quantities so known or observed.

In either case, if the true values of a, b and c were known, each observation of Q would be subject to some error when compared with its value as determined, from the corresponding values of K, L, M, by the law $Q = aK + bL + cM$.

As a, b, c are to be chosen by least squares so as to find values in close agreement with all the observations, we call the errors v_1, v_2, etc.

Then we have

$$aK_1 + bL_1 + cM_1 - Q_1 = v_1$$
$$aK_2 + bL_2 + cM_2 - Q_2 = v_2, \text{ and so on}$$

An equation of this form can be written down for each observation, and is called an *observation equation.*

The total number of such equations must be greater than the number of unknowns a, b, c.

It is required to make the sum of the squares of these errors (which we have agreed to denote by the symbol $[vv]$) a minimum. Hence we first obtain an expression for this sum.

$$\text{Now } v_1^2 = a^2K_1^2 + b^2L_1^2 + c^2M_1^2 + Q_1^2 + 2abK_1L_1 + 2acK_1M_1 + 2bcL_1M_1 - 2aK_1Q_1 - 2bL_1Q_1 - 2cM_1Q_1$$

Writing down a similar expression for each square and adding, we have

$$[vv] = a^2[KK] + b^2[LL] + c^2[MM] + [QQ] + 2ab[KL] + 2ac[KM] + 2bc[LM] - 2a[KQ] - 2b[LQ] - 2c[MQ]$$

where [KK] stands for the sum of the squares of all the observed or known values of K, [KL] for the sum of the products of each value of K by the corresponding value of L, and so on.

To make this a minimum, we differentiate and equate to zero as usual; and as a, b, c are the quantities to be found, we differentiate with respect to those. Further, as they are independent of one another, we differentiate with regard to each separately, taking the other two as constant.

Thus:

$$\frac{d[vv]}{da} = 2a[\mathrm{KK}] + 2b[\mathrm{KL}] + 2c[\mathrm{KM}] - 2[\mathrm{KQ}]$$

$$\frac{d[vv]}{db} = 2b[\mathrm{LL}] + 2a[\mathrm{KL}] + 2c[\mathrm{LM}] - 2[\mathrm{LQ}]$$

$$\frac{d[vv]}{dc} = 2c[\mathrm{MM}] + 2a[\mathrm{KM}] + 2b[\mathrm{LM}] - 2[\mathrm{MQ}]$$

Rearranging these to put the unknowns always in the same order, equating to zero, and cancelling out 2 in each case, we obtain

$$a[\mathrm{KK}] + b[\mathrm{KL}] + c[\mathrm{KM}] - [\mathrm{KQ}] = 0 \quad . \quad . \quad (1)$$

$$a[\mathrm{KL}] + b[\mathrm{LL}] + c[\mathrm{LM}] - [\mathrm{LQ}] = 0 \quad . \quad . \quad (2)$$

$$a[\mathrm{KM}] - b[\mathrm{LM}] + c[\mathrm{MM}] - [\mathrm{MQ}] = 0 \quad . \quad . \quad (3)$$

From these equations a, b, and c can be found. They are called the *normal equations*, and their form enables us to write them down at once, when the form of the observation equation is known, without going through the labour of squaring and differentiating in each case.

Thus take the observation equation

$$a\mathrm{K} + b\mathrm{L} + c\mathrm{M} - \mathrm{Q} = v$$

Imagine each coefficient on the left multiplied by K, and we obtain

$$a\mathrm{KK} + b\mathrm{KL} + c\mathrm{KM} - \mathrm{KQ}$$

Sum these for all the observations and equate to zero, and we obtain the *first* normal equation.

Now multiply each coefficient on the left by L instead of K, and sum, and we obtain the *second* equation, and so on.

Mean Square and Probable Errors.—Where quantities are found indirectly in this way, the rules for calculating the mean square and probable errors are less simple.

In each normal equation there is an *absolute term*, that is, a term which consists of a mere number and does not involve any of the unknowns.

These are denoted above by [KQ], [LQ], etc. If the probable errors are required, then in solving the normal equations we should

retain the above symbols for each of these terms, and find the various unknowns in terms of these symbols.

We thus obtain results in the form

$$a = p_1[\mathrm{KQ}] + q_1[\mathrm{LQ}] + r_1[\mathrm{MQ}]$$
$$b = p_2[\mathrm{KQ}] + q_2[\mathrm{LQ}] + r_2[\mathrm{MQ}], \text{ and so on}$$

Here K is the coefficient of a in the original equation, and we note that p_1 is the coefficient of KQ in the final equation for a. We shall require to make use of this. Similarly L is the coefficient of b in the observation equation, and q_2 is the coefficient of [LQ] in the equation for b, and so on.

Next we must find the mean square error of one single observation.

For this purpose calculate each residual by the formula

$$v_1 = a\mathrm{K}_1 + b\mathrm{L}_1 + c\mathrm{M}_1 - \mathrm{Q}_1, \text{ and so on}$$

Let n be the number of observations, and N the number of unknowns (in this case 3).

Then mean square error of one observation $= \mu$

$$= \sqrt{\frac{[vv]}{n - \mathrm{N}}}$$

Then it can be shown that if μa be the mean square error of a, then $\mu a^2 \times \frac{1}{p_1} = \mu^2$ or $\mu a = \mu\sqrt{p_1}$, where p_1 is the coefficient of [KQ] in the equation for a, as above.

Hence in this case

$$\mu a = \sqrt{p_1} \times \sqrt{\frac{[vv]}{n - \mathrm{N}}}$$

$$\text{similarly } \mu b = \sqrt{q_2} \times \sqrt{\frac{[vv]}{n - \mathrm{N}}}, \text{ and so on}$$

The probable error in each case is found by multiplying these results by 0·6745.

The *weights* of the unknowns are $\frac{1}{p_1}$ and $\frac{1}{q_2}$ respectively.

Example.—As an example take the following observations by the writer for determining the constants of a subtense micrometer on a tacheometer with movable hair :—

No.	D (ft.)	n.	S (ft.)	D — k.	s=S÷(D−k).	s^2.	$n.s$.
1	100	24·786	1	98·32	0·010170	0·00010345	0·25209
2	200	24·680	2	198·32	0·010085	0·00010170	0·24889
3	200	18·261	1	198·32	0·005042	0·00002543	0·09208
4	250	16·964	1	248·32	0·004027	0·00001622	0·06832
5	300	16·097	1	298·32	0·003352	0·00001124	0·05396
6	350	15·479	1	348·32	0·002871	0·00000824	0·04444
7	400	15·023	1	398·32	0·002511	0·00000630	0·03772
8	400	18·174	2	398·32	0·005021	0·00002521	0·09125
		149·464			0·043080	0·00029779	0·88875

The instrument was fitted with a fixed central hair, and a movable hair which could not be brought into coincidence with the fixed one.

In use, the fixed hair was brought to the bottom of the staff by the tangent screw, and the movable hair to the top by the micrometer.

In the table, D stands for the measured distance from the instrument; n for the micrometer reading in each case; S for the staff intercept, and k for the additive constant, which was independently found to be 1·68 feet (*vide* Part I.).

The staff intercept was varied with the distance so as to use nearly the whole length of screw, as the testing of the screw did not form part of this particular set of observations.

We have seen (Part I. p. 350) that if M be the reading which the micrometer would show if the hairs could be made to coincide, the formula for distance is

$$D = \frac{S}{n - M} \times \frac{f}{t} + k$$

The problem is, from the measured values to find the best values for M, and for the multiplying constant $\frac{f}{t}$, which we will call c.

We must rearrange our equation, so as to separate these.

We have $(D - k)(n - M) = c \cdot S$

whence
$$n = c \times \frac{S}{D - k} + M$$

The values of $\frac{S}{D - k}$ have been worked out in the table. We shall call this s. Then

$$n = c \times s + M$$

Each value of n is subject to some error, and if we call these errors v_1, v_2, etc., we have a set of observation equations—

$$c_1 \times s_1 + M - n_1 = v_1$$
$$c \times s_2 + M - n_2 = v_2$$

and so on, where
$$s_1 = \frac{S_1}{D_1 - k}, \text{ and so on}$$

We are to make $v_1^2 + v_2^2 + \ldots$ a minimum.

To write down the normal equations, multiply all through on the left by the coefficient of c (viz. s).

$$\therefore c \times s^2 + \text{M} . s - n . s$$

Then sum, and equate to zero.

$$\therefore c \times [ss] + \text{M}[s] - [ns] = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Next multiply by the coefficient of M, and sum, and we have, as there are eight observations,

$$c \times [s] + 8\text{M} - [n] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Before solving these, we shall verify them by actual differentiation.

We have—

$$v_1^2 = c^2 \times s_1^2 + \text{M}^2 + n_1^2 + 2c\text{M} \times s_1 - 2c . s_1 n_1 - 2\text{M}n_1$$

and so on for other errors.

Hence, summing,

$$[vv] = c^2[ss] + 8\text{M}^2 + [nn] + 2c\text{M}[s] - 2c[sn] - 2\text{M}[n]$$
$$\frac{d[vv]}{dc} = 2c[ss] + 2\text{M}[s] - 2[sn]$$

Equating to zero and cancelling out 2, we get the first normal equation given above. In this differentiation we take M as constant. Now differentiate with respect to M, taking c as constant, and we get the second equation. The equations therefore are—

$$8\text{M} + c[s] - [n] = 0$$
$$\text{and } \text{M}[s] + c[s^2] - [ns] = 0$$

The table shows the calculation of the coefficients. The absolute terms $[n]$ and $[ns]$ are retained in symbolic form, and substituting for $[s]$ and $[s^2]$, we have

$$8\text{M} + 0{\cdot}043080c - [n] = 0$$
$$0{\cdot}043080\text{M} + 0{\cdot}00029779c - [ns] = 0$$

Divide each equation by the coefficient of M.

$$\text{M} + 0{\cdot}0053850c - 0{\cdot}125[n] = 0$$
$$\text{M} + 0{\cdot}0069125c - 23{\cdot}213[ns] = 0$$
$$\therefore 0{\cdot}0015275c - 23{\cdot}213[ns] + 0{\cdot}125[n] = 0$$
$$\text{or } c - 15{,}196{\cdot}5[ns] + 81{\cdot}834[n] = 0 \quad . \quad . \quad . \quad (3)$$

Similarly dividing by the coefficients of c and subtracting, we obtain

$$\text{M} - 0{\cdot}56568[n] + 81{\cdot}834[ns] = 0 \quad . \quad . \quad . \quad . \quad (4)$$

Substituting the values $[n] = 149{\cdot}66$ and $[ns] = 0{\cdot}88875$, from the table, these give—

$$\text{M} = 11{\cdot}819$$
$$c = 1274{\cdot}5$$

To check the work, we calculate the residuals by the formula $v = cs + \text{M} - n$, using the above values of c and M, and the tabulated values of s and n. The results are:

No.	v. +	v. −	v^2.
1		0 003	0·000009
2		0·007	0·000049
3		0·015	0·000225
4		0·014	0·000196
5		0·005	0·000025
6		0·001	0·000001
7		0·004	0·000016
8	0·045		0·002025
	0·045	0·049	0·002546

These indicate clearly that No. 8 is a poor reading, and it would, perhaps, be better to leave it out. The slight error in the sums of the positive and negative errors is due chiefly to the fact that M is not *exactly* 11·819.

From (3) the weight of c is $\frac{1}{15196}$.

From (4), that of M is $\frac{1}{0{\cdot}56568}$.

To find the mean square error of one observation

$$\mu = \sqrt{\frac{[vv]}{n - \text{N}}} \quad \text{(p. 48)}$$

$$= \sqrt{\frac{0{\cdot}002546}{8 - 2}} = \sqrt{0{\cdot}0004243} = \pm 0{\cdot}0206$$

$$\therefore \text{ probable error of one obs.} = \pm 0{\cdot}0139$$
$$\text{and } \mu c = \sqrt{15196{\cdot}5} \times {\cdot}0206 = 2{\cdot}5 \text{ nearly}$$

Probable error in $c = \pm 1{\cdot}7$ nearly, which is about one part in 750.

$$\mu\text{M} = \sqrt{0{\cdot}56568} \times 0{\cdot}0206 = \pm 0{\cdot}0155$$
$$\text{probable error in M} = \pm 0{\cdot}010$$

To conclude the exercise, the values of c and M should be substituted in the equation for D (p. 49), and the results worked out for each distance, to see how far these constants give satisfactory results. This is left to the student.

It will be found that the error is always less than 1 in 400, except for the last reading, where it reaches 1 in 150.

Of course the working here given makes the sum of the squares of the errors in n a minimum. It would have been better to keep to the formula for D, but the problem would have been much more difficult.

Weighted Observations.—We have hitherto supposed that each observation used in the calculation is, so far as is known, likely to be no better or worse than any other. In other words,

that they are all of *equal weight*. But suppose we are dealing, for instance, with a series of angles, of which one has been observed 12 times, another 6 times, another only twice, always with equal accuracy; and suppose the observations in each case represent the mean of these 12, 6, or 2 measurements.

Surely that angle which was observed 12 times should be regarded as more accurately known than that which was observed twice?

Or the observations may be taken with different (not equally sensitive) instruments, by different (not equally skilful) observers, by the same observer in different degrees of health or fatigue, or under different atmospheric conditions, and so on.

In such cases, each observation should have a number attached to it, representing its *relative weight* in the series.

To decide upon the proper value for this number in each case is clearly a very difficult matter, and one attended with much uncertainty.

If a number of similar quantities, like angles or lengths, be each observed several times, and the means taken, the relative weight to be applied to each mean may be taken as simply equal to the number of measurements from which it was derived. Clearly, however, the observations themselves may show that some sets are better than others. In this case, the relative weight of each mean may be found from the formula

$$\text{weight of each mean} = \frac{n \, . \, (n-1)}{[vv]}$$

where n is the number of measurements from which the mean is derived, and $[vv]$ is the sum of the squares of the residuals obtained by subtracting each measure from the mean. The theoretical weight is, in fact, proportional to the reciprocal of the square of the mean square error, which, as we have seen, is $\frac{[vv]}{n \, . \, (n-1)}$ (p. 44).

Thus if several measures of the length of a line be taken say on each of several different days by different observers with different apparatus, each mean could be weighted by this formula. But of course there is still the chance that observer, apparatus, or conditions of work may lead to uncorrected systematic errors greater in some sets than in others; these would not affect the mean square errors, and hence would not be allowed for in weighting by this formula, while it is clear that they would alter the relative values of the different sets.

Hence the question of weighting becomes largely one of

judgment on the part of the responsible heads, and the assigned weights must be regarded merely as giving some uncertain amount of improvement on the results which would follow from assuming all weights equal.

Rules for Weighted Observations.—We shall now suppose, however, that such weights have been assigned to the different observations, and that they are denoted by P_1, P_2, etc.

Then if the observations are *direct*, so that we are simply taking the arithmetic mean, we must multiply each result by its *weight number*, add the results, and divide by the sum of the *weight numbers*.

That is, if M_1, M_2, etc., be the respective measures, M the mean, then

$$M = \frac{P_1M_1 + P_2M_2 + \ldots}{P_1 + P_2 + \ldots}$$

If the observations are indirect, then in forming the normal equations we must write the observation equations thus:—

$$P_1aK_1 + P_1bL_1 + P_1cM_1 - P_1Q_1 = P_1v_1$$

and so on, or generally

$$aPK + bPL + cPM - PQ = Pv$$

Multiply each term on the left by K, and add, and the first normal equation becomes

$$a[PKK] + b[PKL] + c[PKM] - [PKQ] = 0$$

and so on.

In fact, the observation equations may be reduced to equal weight by multiplying throughout each equation by the *square root of the corresponding weight*. Thus if there are a number of observations in the form $Q = aK + bL + cM$, to which weights P_1, P_2, etc., are assigned as above, we may write the observation equations

$$aK_1\sqrt{P_1} + bL_1\sqrt{P_1} + cM_1\sqrt{P_1} - Q_1\sqrt{P_1} = v_1\sqrt{P_1}$$
$$aK_2\sqrt{P_2} + bL_2\sqrt{P_2} + cM_2\sqrt{P_2} - Q_2\sqrt{P_2} = v_2\sqrt{P_2}$$

and so on. These are then of equal weight, and if treated as described on p. 47, the same normal equations will be obtained as by the alternative method just given.

General Remarks.—The above remarks are intended to give the student an elementary idea of the general principles on which observations are adjusted. For a full account of them he must refer to the special works.

It is right, however, to caution the student against, as Mr. Wright says, "the common idea that if we have a poor set of observations good results can be derived from them by adjusting them according to the method of least squares, or that, if work has been coarsely done, such an adjustment will bring out results of a higher grade. . . . The method of least squares is no philosopher's stone: it has no power to evolve reliable results from inferior work."

And again, "The mean square error and probable error of series of observations have been defined as measures of their relative accuracy. With ideal series . . . this is true. . . . But in any actual series, selected at random, we must apply these tests with caution. . . . For example, in levelling, if the same line is run over in duplicate in the same direction, a good agreement may be expected at the several bench-marks where comparisons are made. The mean square error of observation will consequently be small. If the line is levelled in opposite directions experience shows that the agreement would not be so good. The mean square error would be larger than before. We might, therefore, hastily conclude that the first work would give the better result. But . . . it is to be expected that the final result obtained from measurements in opposite directions will be nearer the truth. The conclusion arrived at by trusting to the mean square error alone would be illusory." The discrepancy between the results in this case is, at least partly, due to systematic movements of the ground under the staff and instrument.

Exactly the same remarks apply to the probable error in the length of a base line measured several times by any one method. The probable error may be very small, yet there may be some constant error due to the apparatus employed, or the mode of using it, which will not be detected by theory, though perhaps much greater than the estimated probable error.

Or again, suppose we measure a horizontal angle say 20 times, with a theodolite reading to minutes only. We may get exactly the same value for each of the twenty readings, *because* our instrument is rough, and gives us only the nearest minute. These observations would show a probable error of zero; nor would the method of least squares enable us to find a better value of the angle (though in some cases, not quite so simple, it might *apparently* do so), and yet we would feel quite sure that the true value of the angle was *not* an exact number of minutes.

indeed, such observations will probably give very erroneous results. Even mean sea-level cannot be obtained accurately in this manner. Neglect to take continuous observations is the principal cause of tidal phenomena being pronounced to be anomalous.

An approximation to mean sea-level may be obtained by observations extending over a lunar month, but absolute accuracy can only be arrived at by observations extending over the period stated above.

Assumptions on which Tides are Investigated and Predicted.—Guided by the general principles which have been briefly described, and having to hand the results of *continuous* observations, extending over 369 days 3 hours (25 lunations or lunar fortnights), tides may be predicted with any degree of accuracy. To do so, the following fiction is used. The actual tide curve, as obtained by plotting hourly observations (taken during the period named above), is assumed to be the resultant of a number of tide-waves superimposed, each having different *periods* between successive high waters, and different *ranges*, that is to say, differences of level between high and low water. The range of each *individual* tide is assumed to be constant. The *periods* or times elapsing between the successive high water of each component tide, are also assumed to be constant, and to bear some fixed proportion to the real mean motions of the real sun and moon, such as a solar day, a lunar day, half a solar and half a lunar day, a fortnight, a year, and so on.

The rise and fall of each tide is assumed to obey the law of a harmonic curve or oscillation, a term which will be explained later on.

It is capable of demonstration that the elevations at opposite sides of the earth produced by the attraction of the moon, would be produced to the same degree and to the same extent if the moon were cleft into two equal parts, placed equidistant from the earth and opposite to each other. The moon and ante-moon would transit successively at intervals of twelve lunar hours, producing tides at these intervals. So also for the sun. Then, to produce the effect of diurnal inequality a further moon and sun are imagined, passing the meridian at intervals of twenty-four hours, and so on.

A number of imaginary stars are assumed, each producing a high tide, immediately under it but *not* opposite it. The period of these stars, or their times of apparent revolution, are assumed to be multiples or submultiples of the real mean motions of the sun and moon. Imaginary stars are also assumed, to take account of perturbations produced by the variations of the *rate* of movement of the moon and sun, and of their distances from the earth,

for a survey of 1000 square miles or more, and will afford the means of preparing a map of a considerable district, to any desired scale, possessing sufficient accuracy for all practical purposes.

Arrangement of Stations.—The stations for a triangulation are, perhaps, best arranged in interlacing polygons, as illustrated by the diagram (Fig. 23), which shows the triangulation of Malta.

It will be seen that the area is covered by a series of polygons, each divided into a number of triangles by means of a central station, certain triangles being common to two or more polygons.

Each station should be visible from all the adjacent stations, as shown by the lines on the figure. At some convenient point where the ground is level or otherwise suitable for measurement, a base line is chosen and measured. A separate chapter is here devoted to this measurement, in consequence of its difficulty and importance.

All the angles are then observed, as will be more fully described later, and the remaining sides calculated from these angles, by the well-known formulæ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Correction of Angles.—Though not absolutely necessary, it is desirable to measure the third angle of each triangle. This not only gives a check on the accuracy of the angular measurements, but materially increases the probable accuracy of the work. The three angles will not in all probability sum to 180°, but to some seconds more or less. The *excess* or *defect*, *over* or *under* two right angles, is called the "*summation error*" of the triangle. This should not exceed some prescribed amount dependent on the capacity of the instrument used and the care and skill of the observer.

Before calculating the sides the three observed angles must be corrected by adding to or deducting from each a small angle, so that the corrected angles sum exactly to 180°. In any given triangle there is no particular reason why one angle should be more in error than another, nor is the probable error in any way dependent on the magnitude of the angle, as it is on the length of linear measurements. So, as far as a single triangle by itself is concerned, the summation error may be divided by three, and the correction so obtained applied to each angle in such manner as to make the corrected angles sum to 180° 0′ 0″. Although this simple method of correction suffices for a single triangle, when a considerable number of triangles have to be corrected, a more elaborate method of determining the proper correction to be applied to each angle becomes necessary.

"Equations of Condition" of a Closed Polygon.—In order to ascertain how the necessary corrections may be made, so as to

FIG. 23.

[To face p. 56, Middleton and Chadwick's "Surveying," Vol. II.

obtain the desired result, it will be necessary to examine the "equations of condition" of a closed polygon, composed of several triangles fitted together.

Let ABCDEF (Fig. 24) be any polygon. Let rays be drawn to some point O inside the polygon. Then the polygon may be considered as made up of the triangles AOB, BOC, COD, DOE, EOF, and FOA.

In the first place, it is evident that the three angles of each triangle must sum to 180°, assuming that spherical excess may be neglected.

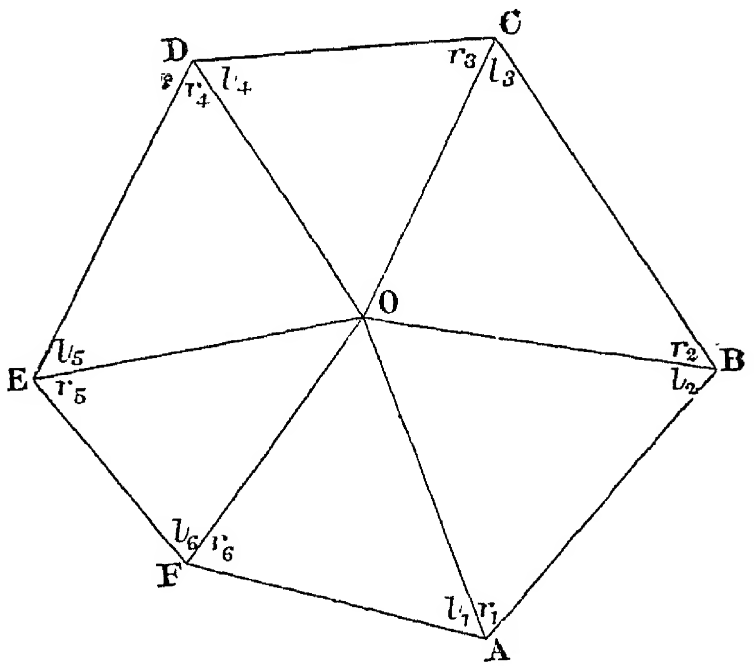


Fig. 24.

This condition expresses the relation between the angles of each triangle, and the equation which results from it is called an *angle* (or *triangle*) equation.

Next we consider the relation between the angles at any one point, the resultant equation being called a *local* equation.

In this case, the sum of the angles at O must be 360°. If the angles are observed in a continuous round with a theodolite (as is usual) and the instrument checks back to zero on the first station, the angles, being obtained by successive subtraction, must sum correctly. Each angle is liable nevertheless to error, like all other angles. If they were observed independently (as with a sextant, for example), they would not in all probability even sum correctly.

Now suppose that, the angles having been corrected to satisfy these two conditions only, the polygon is drawn on paper, starting with any angle AOB (Fig. 25) and continuing the next triangle BOC from the side OB. The triangles will then fit together as in Fig. 25; the angles at O will sum to 360°, and the three angles of each triangle will sum to 180°. But there is nothing to ensure that the final side OA_1 of the last triangle shall be of the same length as OA in the first.

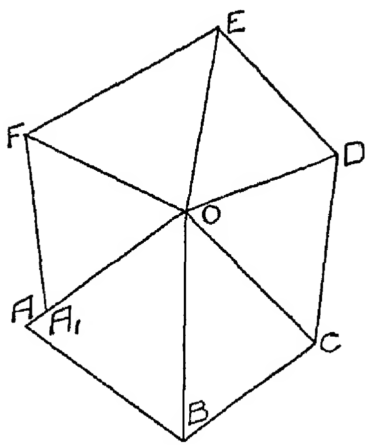


FIG. 25.

For this reason it is necessary to introduce a third condition which must be satisfied.

Now suppose that the observer is at O (Fig. 24). Then the angles of the several triangles at the circumference of the polygon, may be called *right hand* or *left hand* as he regards them in succession. Thus $r_1\ r_2\ r_3\ r_4\ r_5\ r_6$ are *right-hand* angles, and $l_1\ l_2\ l_3\ l_4\ l_5\ l_6$ are *left-hand* angles, each of these being on the observer's *left* as he stands at O and looks down the middle of the triangle to which it belongs.

Now by plane trigonometry

$$\frac{OB}{OA} = \frac{\sin r_1}{\sin l_2} \qquad \frac{OE}{OD} = \frac{\sin r_4}{\sin l_5}$$

$$\frac{OC}{OB} = \frac{\sin r_2}{\sin l_3} \qquad \frac{OF}{OE} = \frac{\sin r_5}{\sin l_6}$$

$$\frac{OD}{OC} = \frac{\sin r_3}{\sin l_4} \qquad \frac{OA}{OF} = \frac{\sin r_6}{\sin l_1}$$

Multiply these equations together,

$$\frac{OB \times OC \times OD \times OE \times OF \times OA}{OA \times OB \times OC \times OD \times OE \times OF} =$$

$$\frac{\sin r_1 \times \sin r_2 \times \sin r_3 \times \sin r_4 \times \sin r_5 \times \sin r_6}{\sin l_1 \times \sin l_2 \times \sin l_3 \times \sin l_4 \times \sin l_5 \times \sin l_6} = 1$$

the products of the sides on the left of the equation cancelling.

Then we have the precept that, *In any closed polygon composed of triangles, the continued product of the sines of the right-hand angles is equal to the continued product of the sines of the left-hand angles.* The equation which results from this condition is called a *side* equation.

The log sines of the several angles may be conveniently taken in lieu of the natural sines.

We thus have the three equations of condition of a perfect polygon :

(1) *The three angles of each triangle must sum to two right angles.*

(2) *The angles at the central point must sum to four right angles.*

(3) *The sum of the log sines of the right-hand angles must be equal to the sum of the log sines of the left-hand angles.*

With Exterior Central Station.—So far the central station has been assumed to be inside the polygon, the case which usually occurs in practice. The same principle applies, with a little modification, to the rarer case where O is exterior to the polygon (Fig. 26).

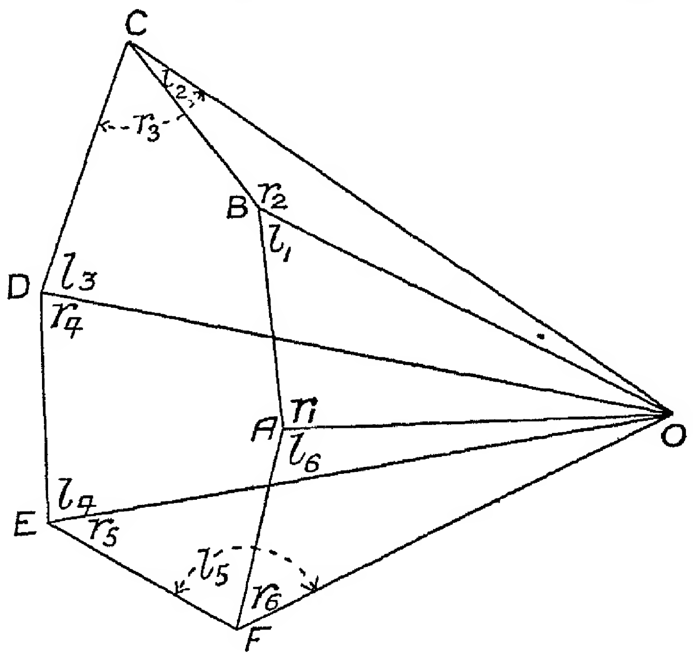


Fig. 26.

Thus the polygon ABCDEF is obtained by taking away the three triangles AOF, AOB, and BOC from the figure OCDEF made up of the triangles COD, DOE, EOF. The first-named triangles may therefore be called *negative* triangles, the last three *positive* triangles. It is unnecessary to repeat the reasoning which is precisely like that of the first case, except that, as a little consideration will show, when O is exterior to the polygon the left- and right-hand angles must be reversed in the *negative* triangles.

Conditions (1) and (3) then remain unchanged, but condition No. (2) must be altered to read—

The sum of the centre angles of the positive triangles must be equal to the sum for the negative triangles.

This case might be of use in surveying a coast line. ADO (Fig. 27) would then be the *negative* triangle.

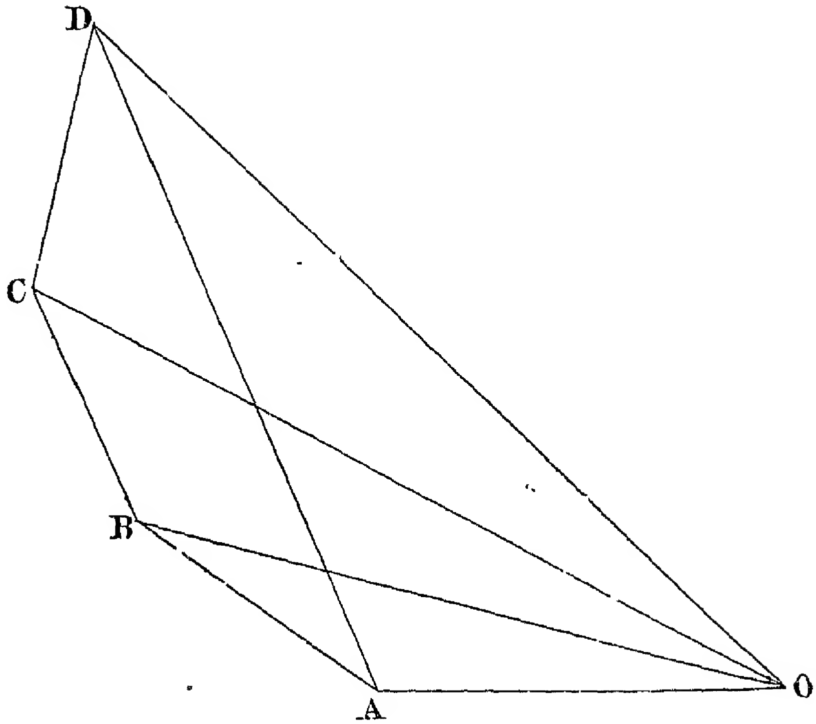


FIG. 27.

Application of the Three "Equations of Condition."—The three equations of condition are applied as follows: The various angles at the different trigonometrical points of a survey having been measured, the three angles of each triangle are computed from the observed angles entered in the "field book."

To avoid confusion in so doing, it is well to prepare a sketch of the network of triangulation to approximate scale, laying off the angles by means of a protractor. This sketch is called the "diagram of triangulation," and an example is given for a portion of the Malta triangulation in Fig. 28, p. 84. Each angle is entered in its proper place on this diagram.

Correction Sheet.—A *correction sheet* is then prepared, of which an example is given in Table A, p. 62. It refers to the same polygons shown in the diagram on p. 84.

The sheet contains columns for the name of triangle, and for the observed *centre*, *left-hand*, and *right-hand* angles; also for the observed sum and total error, for the log sines of left and right angles, and for the *log difference for* 1″ in each case. This is taken out from the tables at the same time as the log sine. It means

the *increase* in the log sine due to an increase of 1″ in the angle, and this when multiplied by the correction to the angle gives the corresponding change in the log sine, for which there is a column headed *corrected difference.* If the angle is greater than 90° the log diff. for 1″ must be entered *minus,* because an *increase* in the angle then means a *decrease* in the sine.

The student should follow out these columns on the sheet.

The corrections are to be applied in such a way as to satisfy the three equations of condition already given.

This can be done in an infinite number of ways, and in large triangulations it is usual to choose that distribution of corrections which makes the *sum of the squares of the corrections a minimum,* according to the principle of least squares.

This involves a large amount of arithmetical labour, however, as will be clear from the small example to be given later, and we may proceed less accurately for a minor triangulation.

Col. Close, in his book on Topographical Surveying, states that for topographical work the total error in each triangle is simply divided equally between the three angles.

This has the merit of simplicity, and it satisfies the triangle equations. But it often adversely affects the *side* and *local* equations, and it is questionable, in the writer's opinion, whether it would not be as well to leave the angles altogether uncorrected.

But inasmuch as the equal division of the error would be the natural course for a single triangle, we may start with this for a first trial, find its effect on the local and side equations, and then judiciously shift corrections from one angle to another until all the equations are satisfied.

Such shifting may be carried out simply according to the judgment of the computer, which is probably a fairly safe course with a skilful mathematician.

The corrections shown in Table A and on the diagram were applied in this way.

In passing from polygon to polygon there are certain triangles common to more than one, and all corrections applied to these triangles in the polygon treated first must, of course, be carried down to the later polygons, and must not again be altered, unless the first polygon is re-corrected.

The only definite principles on which these corrections were applied were that no correction was to exceed 30″; all equations of condition were to be satisfied or very nearly so; and the corrections to common triangles were, as far as could be arranged, to be helpful in correcting all polygons to which they belonged. Triangles Nos. 1 and 2 were previously corrected, and no farther corrections could be made in them.

TABLE A.—CORRECTION SHEET FOR TRIANGLES.

MINOR TRIANGULATION. POLYGON. POINT N CENTRE.

	Centre.		Left.					Right.					Sum of observed angles.	Error.
No. of triangle.	Observed angles.	Cor-rctn.	Observed angles.	Log sine.	Log diff. for 1″.	Cor-rctn.	Cor-rected diff.	Observed angles.	Log sine.	Log diff. for 1″.	Cor-rctn.	Cor-rected diff.		
	° ′ ″	″	° ′ ″			″		° ′ ″			″		° ′ ″	″
2	55 25 30	0	33 06 20	9·7373383	—	—	—	91 28 10	9·9998571	—	—	—	180 00 00	00
1	42 26 30	0	92 36 40	9·9995483	—	—	—	44 56 50	9·8490846	—	—	—	180 00 00	00
7	79 30 30	+ 2	43 28 50	9·8376568	22·2	+ 4	+ 89	57 00 12	9·9236078	13·70	+22	+ 301	179 59 32	+28
8	80 55 32	0	60 04 53	9·9378862	12·1	−18	−218	38 59 35	9·7988068	26·00	+18	+ 468	180 00 00	00
9	51 18 13	− 2	60 18 18	9·9388572	12·0	−20	−240	68 23 57	9·9683761	8·30	−06	− 50	180 00 28	−28
3	50 23 45	0	76 59 15	9·9887020	4·9	− 2	− 10	52 36 42	9·9001148	16·10	+20	+ 322	179 59 42	+18
	360 00 00			$\bar{1}$·4399893			−379		$\bar{1}$·4398472			+1041		
				−379					+1041					
				$\bar{1}$·4399514					$\bar{1}$·4399513					

POLYGON. POINT R CENTRE.

No. of triangle.	Observed angles.	Cor-rctn.	Observed angles.	Log sine.	Log diff. for 1″.	Cor-rctn.	Cor-rected diff.	Observed angles.	Log sine.	Log diff. for 1″.	Cor-rctn.	Cor-rected diff.	Sum of observed angles.	Error.
3	76 59 15	−02	52 36 42	9·9001148	16·1	+20	+320	50 23 45	9·8867540	17·40	00	00	179 59 42	+18
4	53 11 10	+14	71 19 17	9·9765012	7·1	+04	+ 28	55 29 13	9·9159257	14·47	+ 2	+ 29	179 59 40	+20
12	81 16 35	−03	50 49 05	9·8893822	17·15	−05	− 85	47 54 23	9·8704335	19·00	+ 5	+ 95	180 00 03	−03
11	41 55 25	00	62 54 52	9·9495499	10·8	−04	− 43	75 09 45	9·9852719	5·60	+ 2	+ 11	180 00 02	−02
10	38 13 38	−03	68 51 12	9·9697234	8·13	−02	− 16	76 55 13	9·9804111	6·47	+ 2	+ 13	180 00 03	−03
9	68 23 57	−06	51 18 13	9·8923561	16·9	−02	− 34	60 18 18	9·9383572	12·00	−20	− 240	180 00 28	−28
	360 00 00			$\bar{1}$·5776276			+170		$\bar{1}$·5776534			−92		
				+170					−92					
				$\bar{1}$·5776446					$\bar{1}$·5776442					

TABLE A.—*continued.*

POLYGON. POINT S CENTRE.

No. of triangle.	Centre. Observed angles.	Centre. Cor-rctn.	Left. Observed angles.	Left. Log sine	Left. Log diff. for 1″.	Left. Cor-rctn.	Left. Cor-rected diff.	Right. Observed angles.	Right. Log sine.	Right. Log diff. for 1″.	Right. Cor-rctn.	Right. Cor-rected diff.	Sum of observed angles.	Error.
	° ′ ″	″	° ′ ″			″		° ′ ″			″		° ′ ″	″
4	71 19 17	+04	55 29 13	9·9159257	14·47	+02	+ 29	53 11 10	9·9034081	15·75	+14	+220	179 59 40	+20
5	85 24 53	−25	46 50 45	9·8630348	19·70	−23	− 453	47 45 05	9·8693692	19·10	+ 5	+ 95	180 00 43	−43
6	57 34 05	+16	49 13 18	9·8792347	18·15	−20	− 363	73 12 15	9·9810664	6·35	+26	+165	179 59 38	+22
14	43 21 45	−22	88 32 02	9·9998578	0·53	−25	− 13	48 6 38	9·8718265	18·90	+22	+416	180 00 25	−25
13	54 25 37	+22	45 46 50	9·8553216	20·50	−18	− 369	79 47 08	9·9930617	3·80	+21	+ 80	179 59 35	+25
12	47 54 23	+05	81 16 35	9·9949465	3·23	−03	− 10	50 49 05	9·8893822	17·15	−05	− 86	180 00 03	−03
	360 00 00			$\bar{1}$·5083211			−1179		$\bar{1}$·5081141			+890		
				−1179					+890					
				$\bar{1}$·5082032					$\bar{1}$·5082031					

POLYGON. POINT O CENTRE.

No. of triangle.	Centre. Observed angles.	Centre. Cor-rctn.	Left. Observed angles.	Left. Log sine	Left. Log diff. for 1″.	Left. Cor-rctn.	Left. Cor-rected diff.	Right. Observed angles.	Right. Log sine.	Right. Log diff. for 1″.	Right. Cor-rctn.	Right. Cor-rected diff.	Sum of observed angles.	Error.
8	38 59 35	+18	80 55 32	9·9945302	3·37	00	00	60 04 53	9·9378862	12·10	−18	−218	180 00 00	00
9	60 18 18	−20	68 23 57	9·9683761	8·35	−06	− 50	51 18 13	9·8923561	16·87	−02	− 34	180 00 28	−28
10	68 51 12	−02	72 55 13	9·9804111	6·47	+02	+ 13	38 13 38	9·7915375	26·73	−03	− 80	180 00 03	−03
17	91 56 25	−10	26 48 52	9·6542753	41·67	+25	+1042	61 14 40	9·9428415	11·57	−12	−139	179 59 57	+03
16	54 13 38	+12	60 15 28	9·9386529	12·03	+14	+ 168	65 30 58	9·9590786	9·60	−30	−288	180 00 04	−04
15	45 40 52	+02	65 10 00	9·9578626	9·73	−06	− 58	69 09 40	9·9706186	8·02	−28	−224	180 00 32	−32
	360 00 00			$\bar{1}$·4941082			+1115		$\bar{1}$·4943182			−983		
				+1115					−983					
				$\bar{1}$·4942197					$\bar{1}$·4942199					

TABLE A.—*continued.*

POLYGON. POINT Q CENTRE.

No. of triangle.	Centre. Observed angles.	Centre. Cor-rctn.	Left. Observed angles.	Left. Log sine.	Left. Log diff. for 1″.	Left. Cor-rctn.	Left. Cor-rected diff.	Right. Observsd angles.	Right. Log sine.	Right. Log diff. for 1″.	Right. Cor-rctn.	Right. Cor-rected diff.	Sum of observed angles.	Error.
	° ′ ″	″	° ′ ″			″		° ′ °			″		° ′ ″	″
12	50 49 05	−05	47 54 23	9·8704335	19·00	+05	+ 95	81 16 35	9·9949465	3·23	−03	− 10	180 00 03	−03
13	45 46 50	−18	79 47 08	9·9930617	3·80	+21	+ 80	54 25 37	9·9102905	15·00	+22	+ 330	179 59 35	+25
22	85 53 03	+10	41 13 58	9·8189644	24·00	−03	− 72	52 52 45	9·9016569	15·93	+07	+ 111	179 59 46	+14
21	29 48 00	+02	73 31 43	9·9818011	6·20	−01	− 06	76 40 12	9·9881389	4·98	+04	+ 20	179 59 55	+05
20	72 33 17	+09	75 00 15	9·9849522	5·63	+05	+ 28	32 26 10	9·7294554	33·13	+04	+ 132	179 59 42	+18
11	75 09 45	+02	41 55 25	9·8248669	23·45	00	00	62 54 52	9·9495499	10·80	−04	− 43	180 00 02	−02
	360 00 00			$\bar{1}$·4740798			+125		$\bar{1}$·4740381			+ 540		
				+125					+540					
				$\bar{1}$·4740923					$\bar{1}$·4740921					

POLYGON. POINT P CENTRE.

No. of triangle.	Centre. Observed angles.	Centre. Cor-rctn.	Left. Observed angles.	Left. Log sine.	Left. Log diff. for 1″.	Left. Cor-rctn.	Left. Cor-rected diff.	Right. Observsd angles.	Right. Log sine.	Right. Log diff. for 1″.	Right. Cor-rctn.	Right. Cor-rected diff.	Sum of observed angles.	Error.
11	62 54 52	−04	75 09 45	9·9852719	5·60	+02	+ 11	41 55 25	9·8248669	23·45	00	00	180 00 02	−02
20	75 00 15	+05	32 26 10	9·7294554	33·13	+04	+132	72 33 17	9·9795500	6·62	+09	+ 60	179 59 42	+18
19	34 02 22	+16	68 19 12	9·9681379	8·37	+04	+ 33	77 38 02	9·9898052	4·62	+04	+ 18	179 59 36	+24
18	53 52 38	−07	55 51 48	9·9178737	14·28	−02	− 28	70 15 53	9·9737109	7·55	−10	− 75	180 00 19	−19
17	61 14 40	−12	91 56 25	9·9997509	−·72	−10	+ 08	26 48 52	9·6542753	41·67	+25	+1042	179 59 57	+03
10	72 55 13	+02	38 13 38	9·7915375	26·73	−03	− 80	68 51 12	9·9697234	8·13	−02	− 16	180 00 03	−03
	360 00 00			$\bar{1}$·3920273			+ 76		$\bar{1}$·3919317			+1029		
				+76					+1029					
				$\bar{1}$·3920349					$\bar{1}$·3920346					

The Method of Equal Shifts.—But it is distinctly desirable that the corrections should be applied on some principle more or less definite. Perhaps the most simple is what we may call the method of *equal shifts*. That is, any shift which is necessary to satisfy the local equations should be *the same for each triangle of the polygon;* and this should also be true of any farther shift necessary to satisfy the side equations.

In applying this to a single polygon, first correct each centre angle (in pencil) by *one-third* of the total error in the triangle. Sum these corrections, and see how the result affects the sum of the centre angles. We can then easily find by how much we must increase or decrease each correction, the shift being equal in all.

In the first polygon, with centre N, on Fig. 28, taking the results from Table A (p. 62), the errors for the last four triangles are $+28''$, zero, $-28''$, and $+18''$ respectively (see last column).

Dividing these by three, the trial corrections to the centre angles are $+9{\cdot}3$, zero, $-9{\cdot}3$, and $+6{\cdot}0$, of which the sum is $+6{\cdot}0$. But this is to be zero, as the error in the centre angles is zero. Hence each must be *decreased* by *one-quarter* of $6{\cdot}0$, or $1{\cdot}5$, as triangles Nos. 1 and 2 have been already corrected and must not be altered.

Hence the final corrections to the centre angles are $+7{\cdot}8$, $-1{\cdot}5$, $-10{\cdot}8$, and $+4{\cdot}5$ respectively, these being found by deducting $1{\cdot}5$ from the trial corrections.

Now subtract each of these from the total error of the corresponding triangle, and *one-half* of the result will give the trial corrections to the left- and right-hand angles, to be applied *in pencil.*

Thus for the triangle NML (No. 7 in this case)—

$$\begin{aligned} \text{Summation error} &= +\ 28'' \\ \text{Applied to centre angle} &= +\ \ 7{\cdot}8 \\ \hline \text{Remainder} &= \ \ \ 20{\cdot}2 \end{aligned}$$

$$\therefore \text{ trial corrections to left and right angles} = +\ 10{\cdot}1$$

The log differences for $1''$ belonging to the left and right angles of the triangle NML are $22{\cdot}2$ and $13{\cdot}7$ respectively (see Table A).

These are multiplied by the trial corrections (viz. $10{\cdot}1$) to find the corrected differences, viz. 224 and 138.

Following out this process for each triangle we obtain the following results, which would be written in pencil on the sheet, or else on scrap paper :—

No. of triangle.	Left.		Right.	
	Correction (seconds).	Corrected difference.	Correction (seconds).	Corrected difference.
7	+ 10·1	+ 224	+ 10·1	+ 138
8	+ 0·7	+ 8	+ 0·8	+ 21
9	− 8·6	− 103	− 8·6	− 71
3	+ 6·7	+ 33	+ 6·8	+ 112
	Sum =	+ 162		+ 201

Hence the trial corrections have *increased* the log sines of the left angles by 162, and also *increased* those of the right by 201, giving a resultant increase on the right (as compared with the left) of 201–162 or 39.

Now the original sums of the log sines for this polygon are (see Table A, p. 62)—

Left	$\bar{1}$·4399893
Right	$\bar{1}$·4398472
Right too small by . . .	1421

Hence our increase of 39 on the right is in the proper direction, but is much too small. We still require a relative increase of 1421 − 39 = 1382.

Now add up the log differences for 1″ on both sides for those triangles which are being corrected. We have (see Table A)—

$$\begin{array}{ll} \text{On the left} & 22{\cdot}2 + 12{\cdot}1 + 12{\cdot}0 + 4{\cdot}9 = 51{\cdot}2 \\ \text{On the right} & 13{\cdot}7 + 26{\cdot}0 + 8{\cdot}3 + 16{\cdot}1 = 64{\cdot}1 \\ & \qquad\qquad\qquad \text{Total} \quad = 115{\cdot}3 \end{array}$$

This shows that a shift of 1″ *in each triangle* from left to right, or *vice versâ*, will affect the summation of log sines by 115·3. Hence to produce a correction of 1382, still required as above, the number of seconds to be shifted is $\frac{1382}{115{\cdot}3} = 12$ seconds very nearly.

As the left-hand side is to be relatively decreased, we *deduct* 12″ from each left-hand trial correction, and *add* it to the right.

The results are again multiplied by the differences for 1″, to find the corrected differences as here shown.

No of triangle	Left.			Right.		
	Trial correction.	Final correction.	Corrected difference.	Trial correction.	Final correction.	Corrected difference.
7	+ 10·1	− 1 9	− 42	+ 10·1	+ 22·1	+ 303
8	+ 0·7	− 11·3	− 139	+ 0·8	+ 12·8	+ 333
9	− 8·6	− 20·6	− 247	− 8·6	+ 3·4	+ 28
3	+ 6 7	− 5·3	− 27	+ 6·8	+ 18 8	+ 303
		Sum =	− 455			+ 967

The results show a decrease on the left of 455, and an increase on the right of 967, or a relative increase on the right of 967 + 455 = 1422, instead of 1421 required to balance exactly. The discrepancy of one or two units in the last place is negligible.

The student should rule out a correction sheet like that in Table A, and follow out this working for himself. He should have no difficulty in carrying out this process for the whole net, remembering that corrections applied to any triangle in one polygon must be carried forward to the next, and must not be altered unless it is desired to re-correct the first polygon.

The corrections may be taken to single seconds only if desired, of course.

Following a hard-and-fast system like this and carrying on from polygon to polygon, it is inevitable, of course, that certain polygons will be adversely affected by the corrections brought down to it, and the magnitude of the corrections will gradually increase.

In this particular example, if we carry this method through all the polygons in the order shown, we find that the maximum single correction becomes 37″; there are 16 corrections of 15″ or over, and the sum of the squares of all the corrections is about 11,800. These results could possibly have been improved by taking the polygons in a different order—beginning with the worst. Comparing with the corrections in Table A, we find that there the greatest single correction was 30″ (this was fixed as a condition in the adjustment), there are 20 corrections of 15″ or over, and the sum of all the squares is a little more than 11,300. The results, therefore, are about equal on the whole.

Application to Net.—But it is much better (and not very much more laborious) to take five or six polygons at a time, and we shall illustrate this by correcting the whole net shown in Fig. 28 at once.

$$\begin{array}{rr}
 & \text{h} \quad \text{m} \quad \text{s} \\
 & 17 \quad 15 \quad 43 \\
\text{G.S.T. at G.M.N.} = & 13 \quad 34 \quad 35 \cdot 6 \\
\hline
\text{Sidereal interval} = & 3 \quad 41 \quad 7 \cdot 4 \\
\hline
3^h = & 2 \quad 59 \quad 30 \cdot 5 \\
41^m = & 40 \quad 53 \cdot 3 \\
7 \cdot 4^s = & 7 \cdot 4 \\
\hline
\therefore \text{G.M.T.} = & 3 \quad 40 \quad 31 \cdot 2 \text{ p.m.}
\end{array}$$

Equation of time at G.M.N. on that day $= 14^m\ 8 \cdot 1^s$, to be added to mean time, and increasing $0 \cdot 55^s$ per hour

$\therefore$ equation of time at $3^h\ 40^m$ p.m. $= 14^m\ 10 \cdot 1^s$, to be added to mean time

$\therefore$ G.A.T. $= 3^h\ 54^m\ 41 \cdot 3^s$

Time and Longitude—We have seen (p. 163) that the true local time at any station is not the same as the Greenwich time, unless the station is on the same meridian as Greenwich.

As the meridians rotate with the earth from west to east, the heavenly bodies appear to revolve from east to west.

A distant star, as we have seen, completes one revolution in one sidereal day, or 360° in 24 sidereal hours. That is, the distant stars appear to move at the rate of 15° per sidereal hour, and if two stations differ in longitude by $n°$, the time taken for any distant star to move from the meridian of one station to that of the other will be $\frac{n}{15}$ sidereal hours.

Conversely, if the time interval be observed as t sidereal hours, then the difference of longitude will be $15t°$.

If the body observed be the sun, the time interval is generally observed by a clock giving mean solar time. Now the true sun in this interval will not, in general, move at the same rate as the mean sun, and hence may describe more or less than 15° per mean solar hour. The observed interval must be corrected for the change in the equation of time during that interval (because that tells how much the true sun has gained or lost on the mean), and then reduced to longitude at the rate of 15° per mean solar hour.

The station whose meridian is crossed last will clearly be *west* of the other station.

If one meridian be that of Greenwich, and the other be a place $n°$ west of Greenwich, it is clear that the mean sun will transit (*i.e.* cross the meridian) at Greenwich $\frac{n}{15}$ mean solar hours *before* it transits at the second station. That is, the mean solar time at Greenwich will be *ahead of the local mean time* by $\frac{n}{15}$ hours.

the shift on r_3 and r_4 (to satisfy the local equation at R) is $-\frac{0\cdot7}{2}$, or say 0·4 on r_8 and 0·3 on r_4, giving the values shown in the table.

We have now decided the trial values of $n_3 = +4\cdot0$ and $r_3 = +5\cdot6$. The sum of these is $+9\cdot6$. But the whole error in triangle No. III is $+18\cdot0$. Hence the trial value of the remaining correction, b_3, is $18\cdot0 - 9\ 6$, or $+8\cdot4$.

The triangle No. VII has only one angle at a central station. Its trial correction has been settled as $n_7 = +7\cdot4$. The total error in No. VII is $+28\cdot0$. Hence the trial values for the remaining angles L_7 and M_7 are $\frac{28\ 0 - 7\cdot4}{2} = +10\cdot3$ each.

In this way all the trial corrections shown in the annexed table are found and filled in.

We have now to find the shift necessary to balance the side equations for the whole net taken simultaneously. The principle is that all shifts for any polygon are to be equal as before; that when any angle belongs to two polygons and requires a shift in each, the actual shift should be the *mean* of those belonging to the separate polygons, and that all shifts must be arranged so as not to alter the local equation at any point, or the triangle equations.

We now use the following notation. Let x_N be the shift required in the polygon whose centre is at N; x_R in that which has its centre at R, and so on; and we shall agree that these are to be marked *positive* on left-hand angles, and *negative* on *right* throughout.

Then taking triangle No. IX, for instance, the angle N_9 is a left-hand angle for the polygon with centre at R, and hence should have a shift of $+x_R$ in that polygon; but, as a right-hand angle of the polygon with O as centre, it would require a shift of $-x_O$. Hence, the final shift on that angle would be $\frac{x_R - x_O}{2}$, as shown in the table. Similarly, the shift on R_9 is $\frac{x_O - x_N}{2}$; and that on O_9 is $\frac{x_N - x_R}{2}$. The sum of these is zero, and hence does not affect the total on the triangle. The same principle applies to all triangles common to *three* polygons

For triangle No. III, B_3 is a left-hand angle in the polygon with R as centre, and a right-hand angle in that with N as centre. The shift on B_3 is therefore $\frac{x_R - x_N}{2}$. The angle R_3 is a left-hand

Angle.	Trial correction.	Shift.	Final value.	Angle.	Trial correction.	Shift.	Final value.
N_9	− 9·4	$\frac{1}{2}(x_R - x_O)$	−18·2	S_{14}	− 8·1	—	− 8·1
O_9	− 9·3	$\frac{1}{2}(x_N - x_R)$	−15·3	D_{14}	− 8·5	$-x_S$	+11·5
R_9	− 9·3	$\frac{1}{2}(x_O - x_N)$	+ 5·5	E_{14}	− 8·4	$+x_S$	−28·2
Q_{10}	− 1·0	$\frac{1}{2}(x_R - x_P)$	+ 2·8	S_{13}	+ 8·5	$-\frac{1}{2}x_Q$	+10·2
P_{10}	− 1·0	$\frac{1}{2}(x_O - x_R)$	+ 7·8	E_{13}	+13·0	$\frac{1}{2}(x_Q - x_S)$	+21·2
R_{10}	− 1·0	$\frac{1}{2}(x_P - x_O)$	−13·6	Q_{13}	+ 3·5	$+\frac{1}{2}x_S$	− 6·4
P_{11}	− 0·7	$\frac{1}{2}(x_R - x_Q)$	0	Q_{22}	0	—	0
Q_{11}	− 0·6	$\frac{1}{2}(x_P - x_R)$	− 4·4	E_{22}	+ 7·0	$-x_Q$	+10·3
R_{11}	− 0·7	$\frac{1}{2}(x_Q - x_P)$	+ 2·4	F_{22}	+ 7·0	$+x_Q$	+ 3·7
Q_{12}	− 1·0	$\frac{1}{2}(x_R - x_S)$	+ 7·9	Q_{21}	− 3·1	—	− 3·1
R_{12}	− 1·0	$\frac{1}{2}(x_S - x_Q)$	− 9 2	F_{21}	+ 4·0	$-x_Q$	+ 7·3
S_{12}	− 1·0	$\frac{1}{2}(x_Q - x_R)$	− 1·7	G_{21}	+ 4·1	$+x_Q$	+ 0·8
N_3	+ 4 0	$-\frac{1}{2}\cdot x_R$	+ 5·0	Q_{20}	+ 1·2	$-\frac{1}{2}x_P$	+ 6·0
R_3	+ 5·6	$+\frac{1}{2}\cdot x_N$	− 1·4	G_{20}	+12·6	$\frac{1}{2}(x_P - x_Q)$	+ 9·4
B_3	+ 8·4	$\frac{1}{2}(x_R - x_N)$	+14 4	P_{20}	+ 4·2	$+\frac{1}{2}x_Q$	+ 2·6
N_8	− 2·0	$+\frac{1}{2}x_O$	+ 5·8	P_{19}	+ 6·2	—	+ 6·2
O_8	+ 5·3	$-\frac{1}{2}x_N$	+12·3	G_{19}	+ 8·9	$-x_P$	+18·5
L_8	− 3·3	$\frac{1}{2}(x_N - x_O)$	−18·1	H_{19}	+ 8·9	$+x_P$	− 0·7
N_7	+ 7·4	—	+ 7·4	P_{18}	− 8·1	—	− 8·1
L_7	+10·3	$-x_N$	+24·3	H_{18}	− 5·5	$-x_P$	+ 4·1
M_7	+10·3	$+x_N$	− 3·7	J_{18}	− 5·4	$+x_P$	−15 0
R_1	+ 6·4	$-\frac{1}{2}x_S$	+16·3	P_{17}	− 0·7	$-\frac{1}{2}x_O$	− 8·5
B_1	+ 6·6	$\frac{1}{2}(x_S - x_R)$	− 2·3	J_{17}	− 2·7	$\frac{1}{2}(x_O - x_P)$	+ 9·9
S_1	+ 7·0	$+\frac{1}{2}x_R$	+ 6·0	O_{17}	+ 6·4	$+\frac{1}{2}x_P$	+ 1·6
S_5	−14·0	—	−14·0	O_{16}	+ 4·0	—	+ 4·0
B_5	−14·5	$-x_S$	+ 5·3	J_{16}	− 4·0	$-x_O$	−19·6
C_5	−14·5	$+x_S$	−34·3	K_{16}	− 4·0	$+x_O$	+11·6
S_6	+ 7·6	—	+ 7·6	O_{15}	− 5·4	—	− 5·4
C_6	+ 7·2	$-x_S$	+27·0	K_{15}	−13·3	$-x_O$	−28·9
D_6	+ 7·2	$+x_S$	−12·6	L_{15}	−13·3	$+x_O$	+ 2·3

angle in the polygon with N as centre, and should therefore be given a shift of $+x_N$. But it is a centre angle in the polygon with R as centre, and hence should have *no* shift. We therefore make the shift on R_3 equal to $+\frac{x_N}{2}$ only.

Similarly that on $N_3 = -\frac{x_R}{2}$. The same principle applies to

all triangles common to *two* polygons. In triangle No. VII the shift on M_7 is $+ x_N$, and that on L_7 is $- x_N$, while that on N_7 is zero.

The same principle applies to all triangles belonging to only *one* polygon.

All shifts are filled in in the table (p. 70) on these lines, as shown, so that the final value of n_9 is $-9{\cdot}4 + \frac{x_R - x_O}{2}$, and so on.

Once the idea is grasped, the preparation of this table occupies only about half an hour for six polygons.

Finally, the values of the six unknowns, x_N, x_R, etc., required to balance the side equations, are to be found by the solution of six simultaneous equations, one for each polygon.

Taking any polygon, multiply each correction (as expressed by the formula in the table) by the "log difference for 1″" of the corresponding angle (as taken from Table A).

Call all the products *plus* for the left angles, and *minus* for the right. Then equate the algebraic sum to $\Sigma R - \Sigma L$, where $\Sigma R =$ sum of original log sines of right-hand angles for that polygon, and ΣL is the sum for the left angles. The difference must be taken in the order here shown, and proper attention paid to sign.

Thus for the polygon with N as centre (taking the triangles in the same order as in Table A, p. 62), the left-hand angles to be corrected are M_7, L_8, O_9 and R_3, and the log differences from table A are 22·2, 12·1, 12·0 and 4·9, while the formulæ for the corrections (see p. 70) are

$$10{\cdot}3 + x_N\,; \quad -3{\cdot}3 + \frac{x_N - x_O}{2}\,; \quad -9{\cdot}3 + \frac{x_N - x_R}{2}\,; \text{ and } +5{\cdot}6 + \frac{x_N}{2}$$

respectively. The products for these are *plus*, while the corresponding products for the right-hand angles are minus.

$$\begin{aligned} \text{For this polygon, } \Sigma R &= \bar{1}{\cdot}4398472 \\ \Sigma L &= \bar{1}{\cdot}4399893 \\ \hline \therefore \Sigma R - \Sigma L &= - \quad 1421 \end{aligned}$$

Hence we have the equation

$$\begin{aligned} 22{\cdot}2(10{\cdot}3 + x_N) + 12{\cdot}1\left(-3{\cdot}3 + \frac{x_N - x_O}{2}\right) + 12{\cdot}0\left(-9{\cdot}3 + \frac{x_N - x_R}{2}\right) \\ +4{\cdot}9\left(5{\cdot}6 + \frac{x_N}{2}\right) - 13{\cdot}7(10{\cdot}3 - x_N) - 26{\cdot}0\left(5{\cdot}3 - \frac{x_N}{2}\right) \\ -8{\cdot}3\left(-9{\cdot}3 + \frac{x_O - x_N}{2}\right) - 16{\cdot}1\left(8{\cdot}4 + \frac{x_R - x_N}{2}\right) = -1421 \end{aligned}$$

Forming all the equations in the same way and simplifying, we find

$$
\begin{array}{rrrrrrl}
75\cdot6x_N - 14\cdot0x_R - & 10\cdot2x_O & & & & & = -1189 \\
-14\cdot0x_N + 75\ 6x_R - & 11\cdot7x_O - & 15\cdot9x_S - & 14\cdot9x_Q - & & 6\cdot8x_P & = +\ 295 \\
. \quad -15\cdot9x_R & & +120\cdot2x_S - & 3\cdot5x_Q & & & = -2336 \\
-10\cdot2x_N - 11\cdot7x_R + & 102\cdot9x_O & & & - & 34\cdot2x_P & = +2102 \\
-14\cdot9x^R & - & 3\cdot5x_S + & 108\cdot1x_Q - & & 28\cdot3x_P & = +\ \ 15 \\
-\ 6\cdot8x_R - & 34\cdot2x_O & & - & 28\cdot3x_Q + & 107\cdot1x_P & = -1452
\end{array}
$$

The student should verify these equations for himself. All the multiplications involved can be done accurately enough by slide rule. The differences for 1″ have been used to one decimal place only. It will be noticed that most of the coefficients occur twice. Thus in the first equation (for the polygon with the centre at N) the coefficient of x_R is $-14\cdot0$; then this is also the coefficient of x_N in the equation for the polygon with centre R, and so on.

The work of solution is much simplified by the fact that each equation does not contain all the unknowns. It is as well to use logarithms for the solution, with a rough check by slide rule, unless there are two computers.

Solving, we find approximately

$$x_Q = -3\cdot3\ ;\ x_P = -9\cdot6\ ;\ x_R = -1\cdot9\ ;\ x_S = -19\cdot8\ .$$
$$x_O = +15\cdot6\ ;\ x_N = -14\cdot0$$

Substitute these values in the table on p. 70, and we find the "final values" of the corrections as there tabulated.

Finally we insert these values in the correction sheet, and check the balance of local, triangle, and side equations. It may be necessary in this final stage to add or subtract 0·1 second here or there (as we are only working to the nearest tenth), but this will not appreciably affect the result.

This final check is not shown in the book, but the student may work it for himself. The whole correction of six such polygons means about one day's work for a tolerably capable computer.

The maximum correction is $-34\cdot3''$; the number of corrections of 15″ or over is 13, and the sum of all the squares of the corrections is about 9400. Comparing with the results on p. 67, the advantage of simultaneous correction of the whole net is evident.

To correct the same net by the method of least squares would involve the solution of 28 simultaneous equations and at least thirty times the amount of arithmetical work here required.

Example of External Central Station.—As an additional example of correction, we shall take the annexed example (Fig. 29) from another survey with 5-inch theodolite fitted with micrometers.

This may be regarded as a polygon with central station outside (see p. 59). We might choose any station as the central one, but we shall choose B. Thus BCD shall be called triangle No. I; BDA No. II: and ABC, which is the negative triangle, is No III. Thus the angle BCD is C_1 according to our previous notation (p. 68). The error in that angle is c_1, and so on.

The example also introduces several cases of angles over 90°.

The annexed table (p. 74) shows the correction by the methods explained.

The trial corrections to the centre angles are $+6$, -7, $+4$

Now these should not add to zero. The last should be the *sum of the other two*, as the centre angle for No. III triangle is to be the sum of the other centre angles.

Now $+6-7=-1$. To bring this to $+4$ requires a shift of $5''$, or two on each of the first corrections and one on the last. Thus we get $+8$, -5, and $+3$; then $8-5=3$.

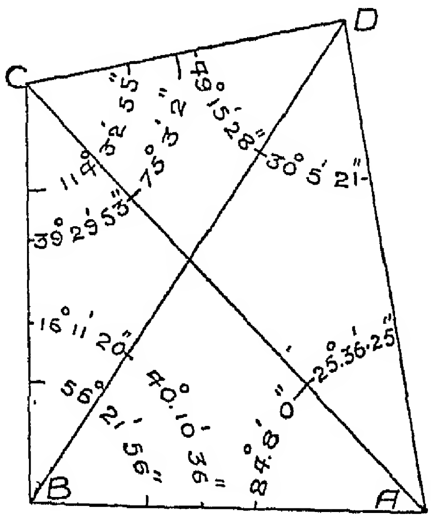


FIG. 29.

The trial corrections to the left angles then become $+4$, -9, and $+4$, giving a difference in log sines of -358; and for the right angles $+5$, -8, $+4$, giving a difference of $+253$.

Thus the total correction to log sines is $358+253=611$, relative increase on the right.

This is in the right direction, but we only required 542. Thus the trial corrections have overdone it by 69.

Now the sum of the differences for $1''$ is 65·0; hence a shift of $1''$ will give us 65, which nearly balances.

The shift is to be in such a direction as to *decrease* the right-hand log sines. Hence we obtain the figures in the table.

The student should carefully work this example through for himself.

Solution by Least Squares.—We shall now work this same example by the method of least squares as an illustration.

We have the following equations :—

(1) $b_1+b_2-b_3=0$. This is a *local* equation, as it gives the relation between the angles at one point B.

(2) $b_1+c_1+d_1=17$, to balance triangle No. 1.

(3) $b_2+d_2+a_2=-22$.

(4) $b_3+c_3+a_3=11$. These are called *angle* or *triangle* equations, as they give the relation between the angles of each triangle.

(5) $18{\cdot}1d_1-7{\cdot}6a_2+25{\cdot}6c_3+9{\cdot}6c_1-36{\cdot}4d_2-2{\cdot}1a_3=542$. This is called a *side* equation, as it is necessary to ensure that each side shall have only one value.

This last equation is formed by multiplying each correction by

Triangle	Centre angle.		Left-hand angle.					Right-hand angle.					Sum observed.	Corr.
	Observed.	Corr.	Observed.	Corr.	Log sin.	Diff. for 1″.	Diff.	Observed.	Corr.	Log sin.	Diff. for 1″.	Diff.		
	° ′ ″		° ′ ″					° ′ ″					° ′ ″	
I, BCD	16 11 20	+8	114 32 55	+5	9·9588548	−9·6	− 48	49 15 28	+4	9·8794707	18·1	+ 72	179 59 43	
II, BDA	40 10 36	−5	30 5 21	−8	9·7001385	36·4	−291	109 44 25	−9	9·9736973	−7·6	+ 68	180 0 22	
III, BCA	56 21 56	+3	84 8 0	+5	9·9977215	2·1	+ 11	39 29 53	+3	9·8034926	25·6	+ 77	179 59 49	
					2·6567148	28·9	−328			2·6566606	36·1	+217		
					−328	36·1				+217				
					2·6566820	65·0				2 6566823				

the corresponding difference for 1″, and equating to the total difference (542) between the sums of the log sines.

The differences for 1″ must have their proper signs for either the right- or left-hand angles, *whichever is too small;* for the angles on the other side the signs are reversed, and the difference between the sums of log sines always has the *plus* sign. Thus, in this case, as the log sines are too small for the *right-hand* angles, the differences for 1″ have their proper signs (as given in the table above) in equation No. 5 for those angles; but the signs are reversed for the left-hand angles. We might equally well, however, always change the signs for right-hand angles, and equate to $\Sigma R - \Sigma L$, as on p. 71.

There are thus *five* equations of condition, and there are *nine* variables, three for each triangle. Hence we choose any *four* of these as *independent variables*, and express all the rest in terms of them by means of the five equations of condition.

We shall choose a_2, b_3, c_1 and d_2.

Then the nine observation equations become—

$$\begin{array}{ll}
(1)\ a_2 = a_2\,; & (2)\ b_3 = b_3\,; \\
(3)\ c_1 = c_1\,; & (4)\ d_2 = d_2 \\
\multicolumn{2}{l}{(5)\ b_2 = -22 - a_2 - d_2, \text{ and } b_1 = b_3 - b_2.} \\
\multicolumn{2}{l}{\therefore\ (6)\ b_1 = b_3 + a_2 + d_2 + 22\,;} \\
\multicolumn{2}{l}{\qquad d_1 = 17 - b_1 - c_1 = 17 - (b_3 + a_2 + d_2 + 22) - c_1.} \\
\multicolumn{2}{l}{\therefore\ (7)\ d_1 = -5 - b_3 - a_2 - d_2 - c_1, \text{ and } c_3 = 11 - b_3 - a_3,}
\end{array}$$

which, substituted in the side equation above, gives

$$-18{\cdot}1(5 + b_3 + a_2 + d_2 + c_2) - 7{\cdot}6a_2 + 25{\cdot}6(11 - b_3 - a_3) + 9{\cdot}6c_1 - 36{\cdot}4d_2 - 2\,1a_3 = 542\,;$$

that is,

$$\begin{array}{l}
-27{\cdot}7a_3 = 350\,9 + 43{\cdot}7b_3 + 25{\cdot}7a_2 + 54{\cdot}5d_2 + 8{\cdot}5c_1\,; \\
\text{or } (8)\ a_3 = -12\,7 - 1{\cdot}58b_3 - 0{\cdot}93a_2 - 1{\cdot}97d_2 - 0{\cdot}31c_1\,; \\
\therefore\ (9)\ c_3 = 23{\cdot}7 + 0{\cdot}58b_3 + 0{\cdot}93a_2 + 1{\cdot}97d_2 + 0{\cdot}31c_1.
\end{array}$$

Each of these equations is in the form $e = m_1a_2 + m_2d_2 + m_3b_3 + m_4c_1 + k_1$, where m_1, m_2, etc., are known coefficients, and e is a general symbol for any one of the corrections, and a_2, d_2, b_3, c_1 are to be found so that the sum of the squares of e shall be a minimum. The values of m_1, etc., are in many cases zero, but this does not affect the form.

The normal equations could be written down at once by the formula on p. 47.

Thus their forms would be—

$$
\begin{array}{l}
a_2[m_1{}^2] + d_2[m_1m_2] + b_3[m_1m_3] + c_1[m_1m_4] \quad [m_1k] = 0 \\
a_2[m_1m_2] + d_2[m_2{}^2] + b_3[m_2m_3] + c_1[m_2m_4] \quad [m_2k] = 0 \\
a_2[m_1m_3] + d_2[m_2m_3] + b_3[m_3{}^2] + c_1[m_3m_4] \quad [m_3k] = 0
\end{array}
$$

and so on, where $[m_1{}^2]$ stands for the sum of all the values of $m_1{}^2$, and so on.

We shall, however, work out the equations from first principles.

We have therefore to work out the sum of the squares of all the above errors, and then to differentiate with respect to each of the four independent variables, and equate to zero in each case.

Squaring the right-hand sides of the observation equations in order, we have

$$
\begin{array}{llllllll}
a_2{}^2 & & & & & & & \\
+ & d_2{}^2 & & & & & & \\
 & + & b_3{}^2 & & & & & \\
 & & + & c_1{}^2 \,. & & & & \\
a_2{}^2 + & d_2{}^2 & & + & 484 + & 44a_2 + & 44d_2 + & 2a_2d_2 \\
a_2{}^2 + & d_2{}^2 + & b_3{}^2 & + & 484 + & 44a_2 + & 44d_2 + & 2a_2d_2 \\
a_2{}^2 + & d_2{}^2 + & b_3{}^2 + & c_1{}^2 + & 25 + & 10a_2 + & 10d_2 + & 2a_2d_2 \\
86a_2{}^2 + & 3\cdot88d_2{}^2 + & 2\cdot5b_3{}^2 + & 0\cdot1c_1{}^2 + & 161 + & 23\cdot6a_2 + & 50\cdot1d_2 + & 3\cdot66a_2d_2 \\
86a_2{}^2 + & 3\cdot88d_2{}^2 + & 0\cdot34b_3{}^2 + & 0\cdot1c_1{}^2 + & 562 + & 44\cdot1a_2 + & 93\cdot4d_2 + & 3\cdot66a_2d_2 \\
\hline
5\cdot72a_2{}^2 + & 11\cdot76d_2{}^2 + & 5\cdot84b_3{}^2 + & 2\cdot2c_1{}^2 + & 1716 + & 165\cdot7a_2 + & 241\cdot5d_2 + & 13\cdot32a_2d_2
\end{array}
$$

$$
\begin{array}{lllllll}
+2a_2b_3 & + 2d_2b_3 & + 44b_3 & & & & \\
+2a_2b_3 & + 2d_2b_3 & + 10b_3 & + 10c_1 & + 2b_3c_1 & + 2a_2c_1 & + 2d_2c_1 \\
+2\cdot94a_2b_3 & + 6\cdot23d_2b_3 & + 41\cdot1b_3 & + 7\cdot87c_1 & + 0\cdot98b_3c_1 & + 0\cdot58a_2c_1 & + 1\cdot22d_2c_1 \\
+1\cdot08a_2b_3 & + 2\cdot29d_2b_3 & + 27\cdot5b_3 & + 14\cdot70c_1 & + 0\cdot36b_3c_1 & + 0\cdot58a_2c_1 & + 1\cdot22d_2c_1 \\
\hline
+8\cdot02a_2b_3 & + 12\cdot52d_2b_3 & + 121\cdot6b_3 & + 32\cdot57c_1 & + 3\cdot34b_3c_1 & + 3\cdot16a_2c_1 & + 4\cdot44d_2c_1
\end{array}
$$

The above summation shows the sum of the squares. The calculation has been worked by slide rule only, as errors are only required to the nearest second.

We now differentiate with respect to a_2, d_2, b_3, and c_1 separately, treating the other three as constant in each case, and equate to zero.

The normal equations which result are—

$$
\begin{array}{r}
11\cdot44a_2 + 165\cdot7 + 13\cdot32d_2 + 8\cdot02b_3 + 3\cdot16c_1 = 0 \\
13\cdot32a_2 + 241\cdot5 + 23\cdot52d_2 + 12\cdot52b_3 + 4\cdot44c_1 = 0 \\
8\cdot02a_2 + 121\cdot6 + 12\cdot52d_2 + 11\cdot68b_3 + 3\cdot34c_1 = 0 \\
3\cdot16a_2 + 32\cdot6 + 4\cdot44d_2 + 3\cdot34b_3 + 4\cdot40c_1 = 0
\end{array}
$$

From these equations we find a_2, d_2, b_3, c_1 to one decimal place :

$$a_2 = -7\cdot9 \,; \quad b_3 = +1\cdot9 \,; \quad c_1 = +4\cdot5 \,; \quad d_2 = -7\cdot7$$

The remaining errors can then be found from the nine observation equations, whence

$b_2 = -6{\cdot}4$; $b_1 = +8{\cdot}3$; $d_1 = +4{\cdot}2$; $a_3 = +5{\cdot}2$; $c_3 = +3{\cdot}9$

These results have not been checked, but, taking them as correct, it will be seen that, to the nearest second, six of the nine agree exactly with the results by the shorter method, while the other three differ by one second.

The sum of the squares of the errors just found is 315; that of those in the table is 318.

Example for exercise: correct by this method the polygon ABROLM of Fig. 28, the first polygon in the table on p. 62.

Remember that triangles Nos. I and II must not be altered. Using the above notation and working by slide rule only, the following answers have been found :—

$r_3 = +0{\cdot}1$; $b_3 = +15{\cdot}3$; $n_3 = +2{\cdot}6$; $l_7 = 10{\cdot}7$; $m_7 = -5{\cdot}0$; $n_7 = +22{\cdot}3$; $l_8 = 12{\cdot}0$; $n_8 = -4{\cdot}2$; $o_8 = +16{\cdot}2$; $r_9 = -1{\cdot}6$; $n_9 = -9{\cdot}1$; $o_9 = -17{\cdot}3$, all in seconds.

Comparing with the results on p. 70, we find close agreement.

Spherical Excess.—In all triangulation work the angles measured are indeed spherical angles, and if observed with absolute correctness, would sum in each triangle to more than two right angles. In the process of adjustment above described, not merely have unavoidable errors of observation been eliminated, but this spherical excess also, for the three observed angles of each triangle have been made to sum to two right angles. The result is that the computed co-ordinates give the relative positions of the points, as though they were on a plane, but coinciding as nearly as possible with their relative positions on the sphere. The distances are all in terms of the base. If the base line were measured at sea-level, "geodetic distances" would be determined, hence, if the base be measured at some elevation above the sea its measured length should be reduced to its "geodetic length" by the rule given on p. 81. This done, all other "distances" will be "geodetic." Beyond this "base reduction to sea-level," no further notice of the spherical form of the globe need be taken in minor "triangulation," as far as horizontal measurements are concerned. (*Vide* also "Legendre's theorem," p. 39.)

In large triangulations the spherical excess is computed, as nearly as possible, from the approximate area of the triangle and the approved radius of the Earth. The triangle equations (p. 57) are then found by equating the sum of the three angles of each triangle to $180° + e$, where e is the spherical excess.

There are, also, many mathematical artifices made use of to shorten the work as far as possible, and enable more triangles to be considered together. The consideration of these is, however, beyond the scope of this work. The student is referred to the books already quoted, and to serial No. 9 of the publications of the United States Department of Commerce, Coast and Geodetic Survey, on the "Application of the Theory of Least Squares to the Adjustment of a Triangulation in Geodesy," by O. S. Adams, published at the Government Printing Office, Washington.

Different Arrangements.—The stations in a triangulation are by no means always arranged in interlacing polygons as has been

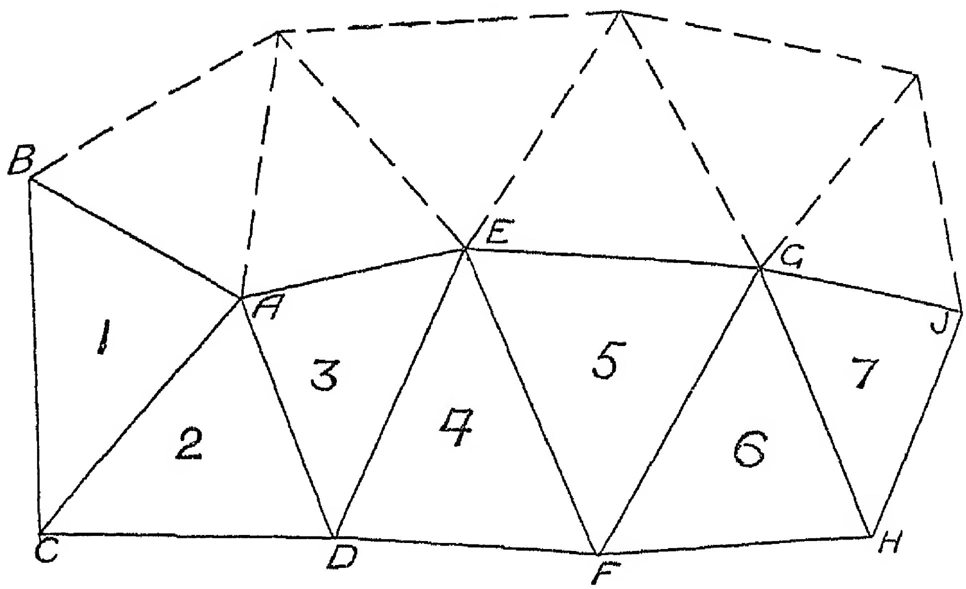


FIG. 30.

here described. Other arrangements, such as that of chains of triangles parallel and perpendicular to meridians, are often used.

The student is referred, for example, to the interesting article on "Surveying" in the "Encyclopædia Britannica" for an account the Triangulation of India.

Base of Verification.—It is usual, as there described, to measure several bases for a large survey. After a triangulation has been corrected as above, the calculated length of any base remote from the original one would seldom agree exactly with the measured length, should no pains be taken to make it do so.

For the purpose of minor triangulation the modification is easily introduced.

Thus in Fig. 30, suppose CB is the initial base, GJ the base

of verification, and that they are connected by a *chain of simple triangles* as shown by the *full* lines.

To calculate from CB to GJ, we must find the sides in order thus: AC from triangle No. 1 (taking AB as known), then AD from No. 2 (taking AC as the known side), and so on.

Let us agree that in any triangle the angle *opposite the known side* shall be called a *left-hand* angle, and that opposite the side to be found is the *right-hand* angle, the third angle being called the centre angle, independently of position. It is clear that in this case in triangle No. 1, A will be the left-hand angle (opposite CB), B will be the right-hand (opposite AC, which is to be found), and C will be the centre angle; for triangle No. 2, angle D (opposite AC) is left, C (opposite AD) is right, and A is centre; and so on.

Now draw up and fill in a correction sheet as before, following this nomenclature.

First apply one-third the error in each triangle to each angle, then sum the log sines of the left-hand and right-hand angles, *including these corrections.* Let ΣR = sum of right-hand log sines, ΣL = sum of left-hand.

Then $\Sigma L - \Sigma R$ should be equal to log AB − log GJ.

If not, let e be the discrepancy, and suppose it is desired that each base shall have exactly its measured value. Then add up the log differences for $1''$ on both left- and right-hand angles. Let s be the sum; then $\frac{e}{s}$ is the number of seconds to be shifted in each triangle, from *left* to *right* (that is *minus* on the left, *plus* on the right) if $\Sigma L - \Sigma R$ is too great (algebraically), and *vice versâ.*

If the triangles form part of a net as shown by the dotted lines, it is best to first correct the net, as previously described, without regard to the bases. Then draw up the correction sheet as above. taking the shortest circuit between AB and GJ. If the discrepancy e found by using the corrected angles is less than that due to the probable errors in the bases, it may be allowed to stand. If not, it can be corrected as above, and these corrections then filled in on the correction sheet for the whole net, and the *remaining* triangles in each polygon (the dotted ones in the figure), then recorrected to make each polygon balance.

In any case, so much of the discrepancy as may be due to the probable or mean-square errors of the bases may, if desired, be allowed to remain uncorrected.

In the Ordnance Survey of the United Kingdom, the measured base at Lough Foyle in the north of Ireland was taken as correct,

and the base on Salisbury Plain, 360 miles away, was used as a base of verification. The calculated length differed from the measured by less than 5 inches.

In minor triangulation bases are measured at much closer intervals than this, of course.

Reduction of Base to Mean Sea-level.—Before computing the sides of the triangles or adjusting the angles as above, the measured base must be reduced to mean sea-level.

Assuming for the moment that the Earth is spherical, and that in a large triangulation the sides are computed by spherical trigonometry starting with the measured base (with angles perfectly observed), it is clear that all sides will be referred to the surface of a sphere having the same centre as the Earth, and passing through the level of the base line.

Now, the horizontal length of a line is the distance between the verticals at its two ends, and it is clear from Fig. 31 that

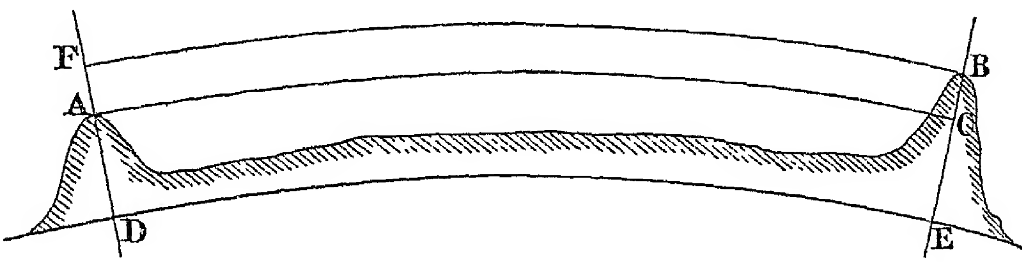


FIG. 31.

this differs with the level at which that distance is measured. Thus FB is longer than AC.

Hence if any side of the triangulation, at a very different level from the base, were measured independently, its measured length would not agree with that calculated.

Moreover, if adjacent countries were surveyed from base lines at different levels, the length of the frontier line would appear greater from the survey of that country whose base was at the higher level, which is, of course, undesirable.

Hence all measured bases are reduced to mean sea-level for large surveys.

Thus if DE (Fig. 31) be at mean sea-level, then either FB or AC would be represented by the length DE.

Distances so reduced are called *Geodetic distances* in surveying. The distances AC and BF are called the "horizontal distances," at the level of A and of B respectively.

To determine the corrections to be applied to reduce a measurement to M.S.L. Let AD $= h_1$, and $r =$ radius of sphere, or radius

of the arc DE, which represents "mean sea-level," and suppose AC is known.

Then

$$\frac{DE}{AC} = \frac{r}{r + h_1}$$

$$= \frac{r}{r\left(1 + \frac{h_1}{r}\right)} = \frac{1}{1 + \frac{h_1}{r}}$$

$$\therefore \ DE = AC\left(1 + \frac{h_1}{r}\right)^{-1}$$

$$= AC - AC \times \frac{h_1}{r} \text{ very nearly, as } \frac{h_1}{r} \text{ is always a small fraction.}$$

Hence the correction to be *subtracted* is

$$AC \times \frac{h_1}{r}$$

Now, as $r = 20{,}000{,}000$ feet (nearly) it is evident that, in most ordinary work, the "geodetic distance" differs from the "measured distance" by a minute fraction only, less than the probable unavoidable error of measurement. Thus if h_1 were 1000 feet, $\frac{h_1}{r} = \frac{1}{20{,}900}$, or less than 1 part in 20,000. This is, however, greater than the probable error of the best measurements, and the correction may be advisable even for a small triangulation.

Computation of the Sides in a Triangulation.—All is now ready for the computation of sides. The attached table (B) is a convenient form, being that used in the topographical survey of India. The angles, etc., are those of the above triangulation and are taken from the diagram (p. 84).

Form of "Computation Sheet."—In copying from document to document, always go back to the primitive document, in this case the "diagram of triangulation." Then by comparing the angles inscribed in the "*computation sheet*" with those of the "*correction sheet*" any error may be detected in time. Next the "*corrections*" are entered from the "*correction sheet*," and being applied, the "*corrected angles*" are entered in the column provided for them. The three angles of each triangle are entered one above the other, placing the angle subtending the side known by previous computation *in the middle*. The several triangles are arranged in chains or circuits starting with the base, or a side known from

previous computation, and closing again upon it, or running from one known side to close on another known side. The best way of doing this will be apparent from the "diagram of triangulation." The log sines of the angles are then looked out to single seconds and written in their proper place. If the surveyor has no one to check him, it will be well at this stage to compare the log sines just obtained with those of the "correction sheet." Adding to or deducting from the latter the "log differences" corresponding to the "corrections" adopted, the two sets of log sines should agree.

Next, the middle log sine is subtracted from the one above it, and from that below it, and the remainders are entered in the column headed "log differences." When this operation is complete for the whole series, the log of the known side is entered in the column "logs of sides," in the middle line, against the angle which is opposite to it. It is then added to the "log differences," and the "sums" are entered in the column "logs of sides." The log of the side common to the first and second triangles is carried into the middle line of the second triangle, and so on till the logarithms of all sides are obtained. Their values can now be looked out.

If the summation of the angles of each triangle, and the central angles, have been corrected to the last second of arc, and if the summation of "log sines" has been corrected to the sixth figure from the left of the sum, then the computed value of last side of the series or chain of triangles should agree with the original value to at least five significant figures from the left. If it do not, then an arithmetical error has been committed. The student may complete the table for himself as an exercise.

Other chains of triangles are now computed in like manner, until the sides of all the triangles have been determined. At this stage of the proceedings the "differences" between two values of any side should be so small as to be negligible.

It will, of course, be clear to the student that the working shown on this form is merely the ordinary solution of each triangle by the formula $\frac{a}{b} = \frac{\sin A}{\sin B}$, and so on, but put into a systematic form.

Computation of Co-ordinates.—The next and final operation is the computation of the "co-ordinates" of the trigonometrical points. The "bearing" of some long line, preferably near to the middle of the survey, is determined astronomically. The point at which the bearing is determined becomes the true origin of the survey, and the "meridian" through that point is the "prime meridian" for

TABLE B.

Computation of Triangles.

Name of triangle.	Angle.	Observed angles.		Corrected angles.	Log sines.	Differences.	Logs of sides.	
		° ′ ″		° ′ ″				
NAM	A	92 36 40		92 36 40	9·9995489	0·1703486	3·6619397	NM
	N	42 26 30		42 26 30	9·8292003	—	3·4915911	AM
	M	44 56 50		44 56 50	9·8490846	0·0198843	3·5114754	NA
		180 00 00		180 00 00				
NBA	N	55 25 30		55 25 30	9·9156025	0·1782642	3·6897396	AB
	B	33 06 20		33 06 20	9·7373383	—	3 5114754	NA
	A	91 28 10		91 28 10	9·9998572	0·2625189	3·7739943	NB
		180 00 00		180 00 00				
NRB	B	52 36 42	+20	52 37 02	9·9001470	$\bar{1}$·9114459	3·6854402	RN
	R	76 59 15	− 2	76 59 13	9·9887011	—	3·7739943	NB
	N	50 23 45	00	50 23 45	9·8867540	$\bar{1}$·8980529	3·6720472	BR
		179 59 42	+18	180 00 00				
BSR	B	55 29 13	+ 2	55 29 15	9·9159286	$\bar{1}$·9394245	3·6114717	RS
	S	71 19 17	+ 4	71 19 21	9·9765041	—	3·6720472	BR
	R	53 11 10	+14	53 11 24	9 0034301	$\bar{1}$·9269260	3·5989732	BS
		179 59 40	+20	180 00 00				

the whole survey as in traverse-surveying. For the purpose of computation it is well to assume an auxiliary origin wholly outside tl : survey, so that all the "meridional distances" and "perpendiculars" will be of the same name. If the "diagram of triangulation" has been made to scale it will be easy to see the value that r st be assumed for the "co-ordinates" of the auxiliary origin to ε that end.

r the known "bearing" of the primitive side, a ries of c verses are set up, which include every point on the survey. rings" of the sides are now computed from the corrected f the "diagram of triangulation" by the precept given u erse surveying, working to single seconds. This operation wı by inscribing the "corrected angles" in red ink on gulation. As the "corrected angles" are in every case computable as for a closed pologon there should be no angular error. Next, the "northings," "southings," "eastings," and "westings," are computed in the usual manner, excepting that, as single seconds are to be worked to, the computation should be made *logarithmically* and *not* with the *traverse table.*

The error in summation of N. S. E. W. should be exceedingly small, so much so that it should be negligible, and the "co-ordinates" are computed in the same manner as for ordinary traverse surveying, except that the correction columns are omitted.

Arrangement of Sheets for Plotting.—The next step is arrange the sheets on which the survey is to be plotted. The "diagram of triangulation" will be useful for this purpose. First lay off the "prime meridian," the "auxiliary meridian," and the "origin." Then draw a number of squares or rectangles, so that the sides represent a round number of feet according to the scale that is to be used. Each sheet should have at least three trigonometrical points upon it. It is in every way desirable that the sheet should not be too large. The smaller the sheet the greater the number of draughtsmen that can be employed on the plan at once. It is well to provide a margin or overlap round each sheet say 0·1 foot wide, common to the adjacent sheets. This will be found exceedingly handy when filling in detail. It is moreover most convenient when several sheets have to be consulted together. A point on the common margin is also most useful when making a measurement involving two sheets. The sheets are prepared and the points plotted in the manner indicated for traverse surveying.

Limit of Accuracy attainable under Different Conditions discussed.—Before describing the process of triangulation it will be well to examine superficially the limit of accuracy attainable under different conditions, and the effect of the form of the triangle thereon.

Let A and B be two points on the meridian, C a third point at a distance from it (*vide* Fig. 32). Let AB be 10,000 units in length.

At A and B angles are measured to C. The angle at C is not supposed to be measured. For simplicity in calculation it will be assumed that the angles at A and B are equal, and that the angle at C is obtained by deducting the sum of A and B from 180°.

Now every angle, however measured, is liable to some error, smaller or greater, according to the power of the instrument and the care employed in using it. In the present case let the limit of error be ± 30″. That is to say, if an angle measured say 49° 32′ 30″, it might in reality be anything between 49° 33′ 0″ and 49° 32′ 0″.

The rays from A and B to C may therefore be regarded as two rods pivoted at A and B, and capable of moving through an angle of 30″ on each side of the true angles to C, at which point they intersect.

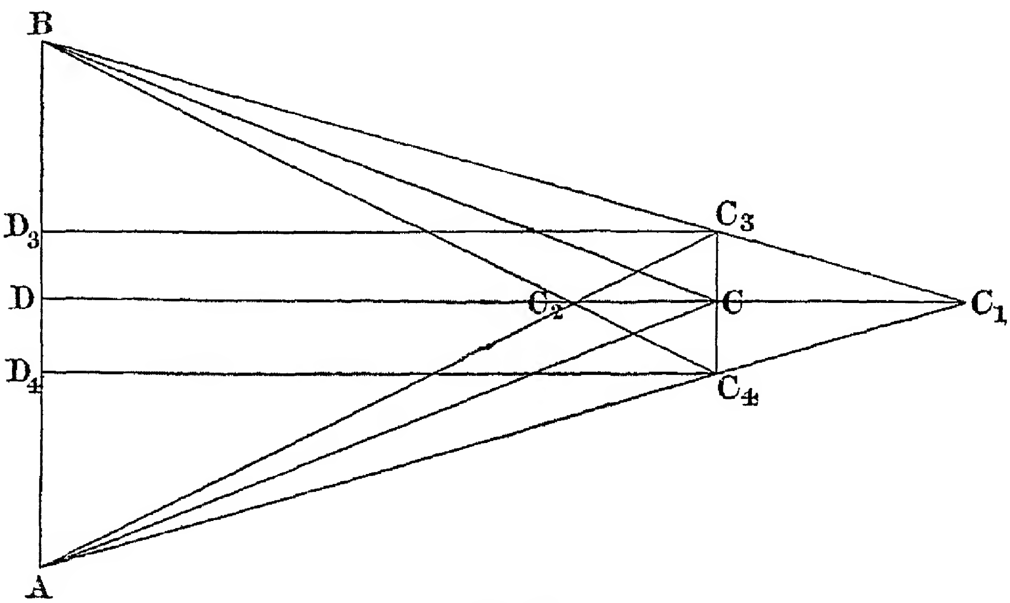


FIG. 32.

It is evident, however, that as each angle is liable to any error up to 30″, the true position of the vertex may be anywhere within the area $C_1C_2C_3C_4$.

Draw C_1D through the points C_1, C, and C_2 perpendicular to AB, also C_3D_3 and C_4D_4. Then DC is the *apparent* "perpendicular distance" of C, and AD the apparent "meridional distance." But the real "meridional distance" may be anything between AD_4 and AD_3, and the "perpendicular" anything between DC_2 and DC_1. The line $CC_3 = CC_4$ may be called the *range of uncertainty* in the "meridian," and $CC_1 = CC_2$ (approximately) the *range of uncertainty* in the "perpendicular." The figure $C_1C_3C_2C_4$ may be called the area of uncertainty. We may also call the ratio $\frac{CC_1}{DC}$ or $\frac{DD_3}{AD}$ the *relative uncertainty* or *rate of probable error* in the "perpendicular" and "meridian" respectively.

Effect of Errors of ± 30″ on Various Isosceles Δ's.—The following table gives the actual and relative errors for various values of the angles observed at the extremities of the base of an isosceles triangle. The base is supposed to be ten thousand units in length.

TABLE SHOWING DIFFERENCES IN THE POSITION OF THE VERTEX OF ISOSCELES TRIANGLES OF DIFFERENT PROPORTIONS, DUE TO A VARIATION OF ± 30″ IN EACH OF THE ANGLES ADJACENT TO THE BASE OF 10,000 FEET.

A and B angles at base.	C angle at vertex.	CC_3 and CC_4 difference in meridional distance due to a variation of ± 30″ in lateral angles.	Difference per 1000 in length of AB.	CD perpendicular distance C from A.	CC_1 and CC_2 difference in perpendicular distance due to a difference of ± 30″ in lateral angles.	Difference per 1000 in length of CD.
° ′	°					
87 30	5	8·344	1·668	114,518·00	383·49 380·95	3·348 3·326
85 00	10	4·188	0·837	57,150·00	95·89 95·57	1·672 1·679
82 30	15	2·809	0·562	37,978·77	42·70	1·124
75 00	30	0·581	0·290	18,660·25	10·85	0·581
67 30	45	0·411	0·205	12,071·06	5·81	0·412
60 00	60	0·335	0·168	8,660·22	2·91	0·336
52 30	75	0·301	0·150	6,516·13	1·96	0·301
45 00	90	0·291	0·145	5,000·00	1·45	0·291
37 30	105	0·301	0·150	3,836·63	1·15	0·301
30 00	120	0·335	0·168	2,866·75	0·97	0·336
22 30	135	0·411	0·205	2,071·07	9·85	0·412
15 00	150	0·581	0·290	1,339·74	0·77	0·581
7 30	165	2·809	0·562	658·26	0·74	1·124

An inspection of this table shows that the *actual*, as well as the *rate of*, *error* is least when the angle at the vertex is a right angle. As the angle varies from a right angle, the *actual* and *the rate of error* increases slowly at first, then more and more rapidly, tending to become infinite as it approaches zero or 180°.

From this we may see that between

$$C = 25° \quad \text{and} \quad C = 155°$$

the rate of error due to an error of ± 30″, in the angles A and B, is always less than 1 to 1000, which has been taken as the error of good ordinary chaining.

We likewise see the advantage of measuring the third angle C. When this is deduced from A and B its value may vary between the limits $C_2 = C + 1' \ 0''$ to $C_1 = C - 1' \ 0''$. But if C be measured we shall obtain its value to within ± 30″, so that the true position can be neither at C_1 nor C_2, but at some point nearer to C.

It is obvious that similar conditions obtain when the triangle is scalene.

So far we have assumed that A and B are fixed points, and that AB is fixed both as to direction and length. But when a number of triangles are built up as a network of triangulation, such as ABCDE (Fig. 33), every angle is equally liable to error, and also every side, except some one primitive side, such as AB, whose direction and length are assumed to be fixed.

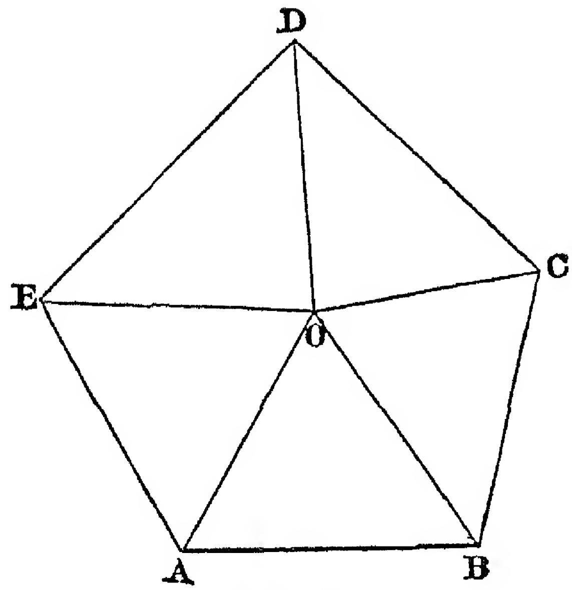


Fig. 33.

Each triangle, therefore, may be considered as being composed of three slender rods pivoted at their middle points (Fig. 34), and which may be moved through a certain small angle representing the probable error in the measurement of any angle. The intersections of these rods in their limiting positions mark out the three areas of uncertainty, within each of which the true position of the point must lie.

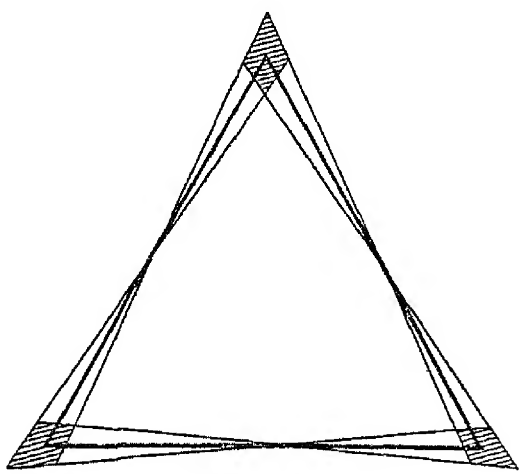


Fig. 34.

Equilateral Δ's the Best.—When the triangle is equilateral the three areas of uncertainty are equal. It is true that an angle of 60° does not give the smallest area of uncertainty. But if one of the angles of a triangle exceeds 60°, one or both of the others must be less than 60°. An inspection of the above table shows that the rate of error increases much more rapidly as the angle becomes less than 60° than it decreases as the angle becomes greater than 60°. Consequently, an equilateral triangle is that in which a given error in each of the three angles produces on the average of all three sides the least error, and therefore the

surveyor should endeavour to make the triangles of a network as nearly equilateral as possible.

Method of conducting a Triangulation.—Turning now to the method of conducting a triangulation. The first step is to select the position of a base line. For this purpose a flat and unbroken piece of country is required. The length of the base line should not be less than about one-third of the average length of a triangle side. A relatively short base measured with great care under favourable circumstances, is preferable to a longer base measured with less precision under less favourable circumstances. It is obviously easier to find a place where 1000 feet can be measured correctly than one where 3000 feet can be set out free from obstructions.

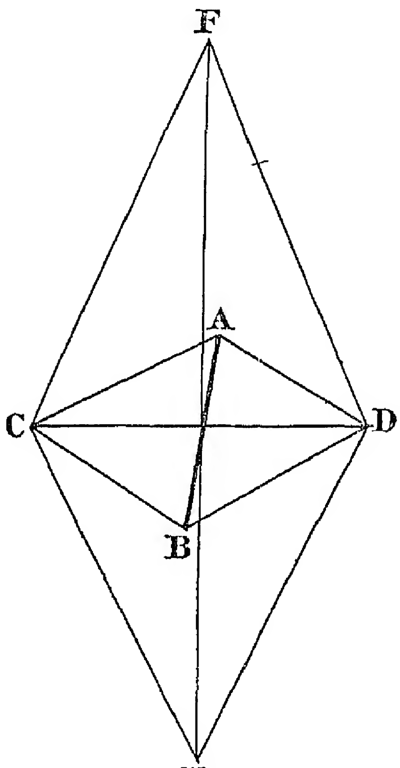


FIG. 35.

Moreover, it is easy to obtain a secondary and longer base by triangulation. Let AB (Fig. 35) be the measured base. From its extremities let angles be taken to C and D. Then, without involving any ill-conditioned triangles, the line DC may be made about double AB. Then in like manner from D and C observations can be made to E and F, so that EF, the extended base, may be about four times the length of the original base.

It is better to extend the base by means of four triangles than by one, for if DC be about twice AB, and FE twice DC, then the angles at C, D, E, and F would be about 53°, whereas in the single operation the vertical angle would be only about 28°, thus giving a far lower degree of accuracy for the single triangle than for any one of the four. The errors, moreover, of the four triangles will not always be in the same direction. They will tend more or less to compensate each other, and thus the probable error of a distance, determined by a series of four well-conditioned triangles, will be less than that determined by one ill-conditioned triangle. With care, the length of the "extended" or "working base" may be determined with an accuracy *nearly* if not *quite* equal to that of the original base.

Even if the surveyor does not possess the means of measuring a base with the highest accuracy, triangulation is still a valuable

method. When a network of triangles is corrected in the manner above described, the *relative* positions of the sundry points will be determined with great precision. The actual distances from point to point will be determined in terms of the measured base, for if the base be measured too long, then all distances will be relatively too long, and *vice versâ*. The effect of a slight error in the measurement of the base is not a distortion of the plan, but amounts to its being drawn to a slightly different scale to that intended. If an error of one part in 2500 were made in the base measurement, the result would be that a plan whose scale was intended to be $\frac{1}{2500}$ might be, in reality, $\frac{1}{2499}$ or $\frac{1}{2501}$, an unimportant difference, and inappreciable on paper.

Suitable Size of Triangles discussed.—The question of the *proper distance between trigonometrical points depends largely on* local conditions, such as whether the formation of the ground be favourable or not. The best average length of triangle sides may be discussed on the following lines.

The positions of the trigonometrical points, as determined by the triangulation, must be considered to be accurate. They must not be re-adjusted, but all subsequent detail field-work must be corrected so as to agree with the trigonometrical points.

The adjustment of chained lines or traverses should never exceed a quantity perceptible on the plan. Hence the length of the triangle side depends upon the scale of the plan to be produced, and on the probable accuracy of the chain or traverse-surveying. Suppose that the plan has to be $\frac{1}{2500}$, and that the probable error of chaining or traversing is estimated at 1 per 1000. Now 0·001 of a foot (0·012 inch) is about the least dimension that the draughtsman can work to. With the scale $\frac{1}{2500}$ adopted, this would represent $2\frac{1}{2}$ feet on the ground. Then the length of the triangle side should be such that, when measured with a chain or estimated from a traverse, an error exceeding $2\frac{1}{2}$ feet cannot accumulate between its extremities. As the error of chaining has been assumed to be 1 per 1000, the average distance between the trigonometrical points should not exceed 2500 feet, say half a mile. If the plan were to be on the scale $\frac{1}{10000}$ (6 inches to 1 mile nearly), then the error on the paper of 0·001 foot would represent 10 feet on the ground. With the same error in chaining, the length of side might be 10,000 feet, say two miles.

Again, in a broken and rugged country where chaining would be inaccurate, the trigonometrical points should be more numerous than in an open, smooth country, where the best work can be expected.

Generally a side of from three-quarters of a mile to a mile in

length is convenient. If much shorter, a very slight displacement of the signal observed to causes inconveniently large errors. If plans on a large scale are to be prepared, for which according to the principles laid down more numerous points would be desirable, it would be preferable to resort to "intersected" subsidiary points, determined in a manner hereinafter described.

Even were the plan proposed in the first instance to be on a scale of $\frac{1}{50000}$, it would be desirable to work with a degree of accuracy that would allow of the survey being plotted to a larger scale at some future date.

The points should be fixed and marked in a permanent manner, and it must be remembered that, in the case of triangulation, permanence is of the greatest importance.

Whilst locating points, it is desirable to prepare a skeleton plan to an approximate scale showing their positions. Such a plan will prove most useful as the work proceeds. For this purpose a compass, pocket-sextant, plane-table, or even a small theodolite (using it as a compass) may be employed. For measuring distances approximately, the telescope of the theodolite may be provided with subtense wires, visually subtending 1 foot at 200 feet. Then by using a staff 16 feet long boldly graduated to feet, a rough approximation to distances up to 3000 feet may be obtained, and information obtained that will often be valuable (especially if a staff-holder has to be sent to put up a mark). A Watkins or Weldon range-finder may be useful, and a powerful binocular is almost essential.

It is not possible to lay down any fixed rule of procedure for the location of points. Much depends on the nature of the country, and the facilities for getting from place to place.

To locate points properly is always laborious, especially when new to the work and the eye is not habituated to judge distances. The assistance of an intelligent foreman chainman is most valuable.

Trig. Points should be selected by the Chief Surveyor.—The selection and location of points is so important that it should be performed by the chief surveyor himself. The observation of the angles may be reduced to a matter of routine, and may be effected by subordinate surveyors. Thus in an extended survey, the chief surveyor locating points may keep several assistants fully employed in observing angles.

The value of the trigonometrical points does not cease with the completion of the survey. They are of the utmost use when the plans, on account of new constructions, have to be corrected up to date. Surveys for new roads, railways, and the like should be

connected to trigonometrical points as well as those of land granted or sold. If the survey be one conducted by Government, a law should be provided giving power to mark survey points, making it a punishable offence to remove, destroy, or obliterate them. If, on account of the construction of a building, a survey point must be removed, then notice should be given to the officer in charge of surveys, who will, if necessary, fix a new mark near to the old one. Moreover, all licensed surveyors should be bound by law to connect their work to established survey points. This being done, and copies of their surveys and traverse-sheets being deposited with the officer in charge of the Government Surveys, a complete map may be compiled, *pari passu* with the occupation of land.

The survey points should therefore be marked in the most permanent manner possible. In addition to this, measurements should be made to fixed and permanent objects. These should be recorded on sketches, showing the measurements on plan, and the elevation and appearance of the objects to which the measurements were made. These sketches should be copied into a book kept for the purpose, and accompanied by a full verbal description. Not only should the points be marked in a permanent manner, but means should be provided for their recovery, if they be lost or obliterated.

Masonry pillars are generally useful as survey marks. They should be founded deep enough to be below the range of surface disturbances due to tillage. If made high enough to serve as a stand for a three-screw theodolite, it would be convenient. The funds at the surveyor's disposal will, however, rarely suffice to erect such high pillars of sufficiently substantial construction to resist the rubbing of cattle and other destructive tendencies.

Signals for Observing to.—The surveyor will therefore have to rest satisfied with marks but little, if at all, above the level of the ground, erecting temporary signals when observing. If there be rock on the surface, a permanent mark may be made by drilling a hole and driving in an iron plug. Or the hole might be filled with Portland cement, a nail being inserted on the top to mark the exact centre. To protect the mark and indicate its position a cairn of stones may be erected over it, and this will serve to hold up the observing signal.

If no objects in the immediate neighbourhood of the trigonometrical point be available for fixing measurements, it will in some cases be possible to fix by cross bearings, each being determined by two objects in line. Thus, for example, the line joining the left-hand angles of the buildings A and B (Fig. 36) might inter-

sect that joining the right-hand angle of C and the left of D in some point X not far from the trigonometrical point O. Then by four measurements a, b, c, d, the point O would be fixed. Sketches should be made, showing the appearance of the objects when in line.

The trigonometrical points, having to be so low as to permit of the theodolite being placed over them, will not be visible from a distance, hence some temporary mark or signal must be erected to observe to.

If the distances be not great, an ordinary ranging rod answers as well as anything else. If labour be cheap, nothing can be better than rods of this kind, each held

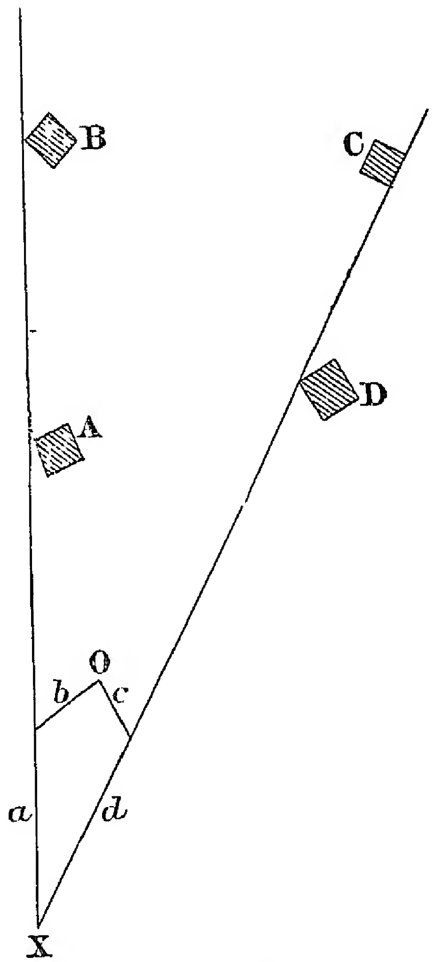


FIG. 36.

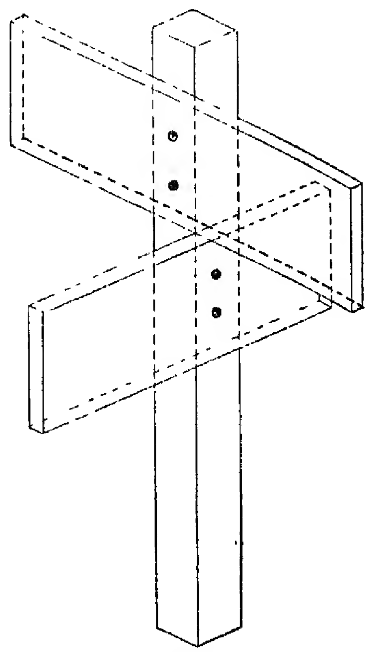


FIG. 37.

on the mark by a man. If the staff-holder stands heels together behind the mark and facing the observer, with the point of the rod on the centre of the mark, and holding it between the palms of his hands in front of his nose, perpendicularity will be secured.

Flags are of little use as signals. If there be wind enough to display the flag, the staff is apt to be pulled sideways. If there be no wind (the best condition for observing), the flag hangs down and is useless.

It will generally happen that semi-permanent signals, fixed so as to dispense with the attendance of a man, have to be used. A good form consists of a piece of $1\frac{1}{2}$-inch to 2-inch scantling (Fig. 37), as straight as possible, to the top of which two pieces of board are nailed at right angles to each other, as shown in the sketch. A coat of whitewash over the boards and rods makes the signal very distinct with most backgrounds. Sometimes, however, it may be well to paint the signals red for a green background, black for a pale grey, such as ploughed land. The signal may be fixed over the mark, either with a small pile of stones or by using three struts, of similar scantling to the signal staff.

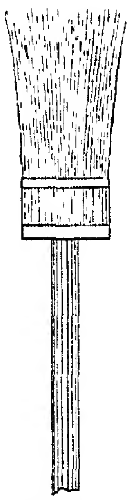


Fig. 38.

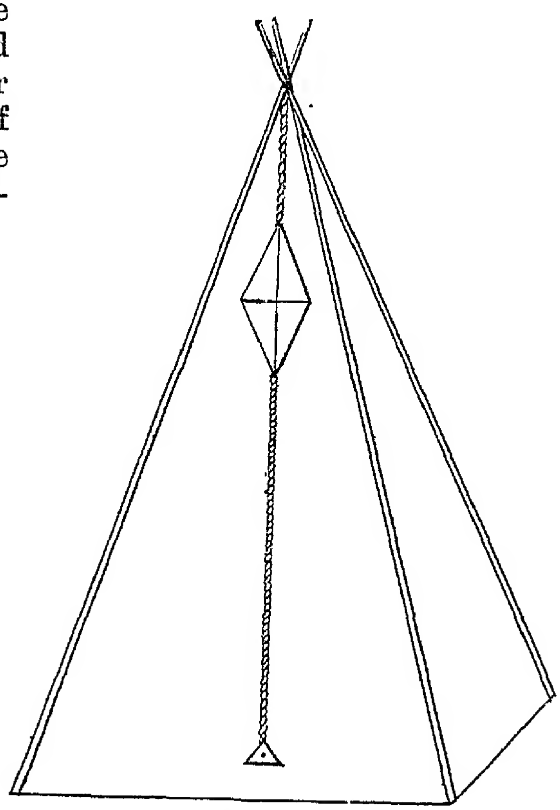


Fig. 39.

A very good signal is made by lashing a number of twigs to a pole or scantling, after the manner of an ordinary besom (*vide* Fig. 38). In lieu of ordinary twigs, the writer has used a bunch of bright-coloured flowers, often found in tropical forests, which made a most conspicuous signal against a green background.

Admiral Belcher in his treatise on Marine Surveying has suggested a form of signal which would appear to be useful (*vide* Fig. 39). The signal consists of a rope, stout enough to be visible and stretched with a heavy weight, suspended from a tripod consisting of three poles lashed together. The weight may be a basket or bucket, with a stout spike fixed to the bottom, and

filled with stones, earth, or water, obtained on the spot. The rope will in any case be perpendicular. If blown aside by the wind, the displacement at the top will be slight, probably less than the displacement of an ordinary signal. Any oscillation may be bisected by the wire of the theodolite. If the point of the spike be not exactly below the centre of gravity of the basket or bucket, correct centring may be effected by spinning it, and moving the legs till the point describes a circle having the mark for a centre. The rope might be painted with different colours to make it conspicuous, or a double cone of painted canvas might be attached to it.

For great triangulation surveys, luminous signals of some kind are generally employed, such as the heliotrope vase-lights, etc. For an account of these, the student is referred to Thuillier and Smyth's "Manual of Surveying for India"; the article "Surveying" in the "Encyclopædia Britannica"; etc.

Observing Angles by "Repetition" and "Reiteration."—The adjustment of the theodolite and its general use have been already described. It will be assumed, therefore, that before commencing the observation of angles, it is practically in perfect adjustment. There are two methods of observing angles, by which more correct values can be obtained than is possible with one single measurement. These are "*repetition*" and "*reiteration*." These have been described in Part I. of this work.

For the purposes of a minor triangulation, "repetition" is far too laborious. Amply sufficient accuracy may be obtained by "reiteration." In observing by "reiteration" the leading vernier is set approximately to zero, and the plates are clamped together. The whole instrument is turned round, and some one signal or mark, called the "referring object" or R.O. is bisected. All the verniers are then read and recorded. Then, the limb remaining stationary, the several signals are bisected in succession, and the reading of each successive point round the circle is recorded, after the manner of "bearings." Lastly the "referring object" is again bisected. This reading should coincide with the first. If it does not, then some slip or torsion of the stand has taken place, or the referring signal has moved. If the second reading of the "referring object" differs seriously from the first, the round is worthless and must be re-observed. If not, both readings should be recorded and their mean used in subsequent calculation. The reading of the "referring object" must not be assumed to be zero.

After this round, the "face" should be reversed (see Part I.).

If the theodolite be provided with three verniers it will not be necessary to alter the position of the graduated arcs. Suppose

that the first round were taken "face left" with the leading vernier A set to zero, B to 120°, and C to 240°, then transit the telescope and turn it through 180°, so as to bring it "face right," and again bisect the zero station: A vernier will read 180°, B 300°, and C 60°. Thus each angle will be read on six parts of the limb.

If on the other hand there be two verniers only, then the zero station should be bisected for the second round, with the leading vernier set to some other angle, preferably 90°. The automatic zero and right angle theodolite (p. 19) is useful for this.

It is well also to observe the second round of angles in the opposite direction to the first. That is to say, if the telescope has been, on the first round, directed to the successive points from left to right with the hands of the clock, then for the second round it should be moved in the opposite direction. This tends to eliminate slips of the clamps. Moreover, it has been found that the action of the moving sun on a tripod causes the latter to gradually twist in one direction (assuming that the sun shines continually).

Mr. Wright quotes an example from the U.S. Lake Survey in which the twist of the centre post carrying the theodolite was at the rate of 1 second of arc per minute of time (*vide* "Adjustment of Observations," p. 89).

Now it is clear that the above method of measurement tends to eliminate the effect of this, so long as it be uniform.

For other precautions, see *Reiteration*, Part I.

For the purposes of a "minor triangulation," two "reiterations" usually suffice, provided that the instrument be well graduated and solidly constructed.

The mean of all the readings of all verniers is then worked out to the nearest second, and the angles are obtained by successive subtraction.

It is scarcely necessary to say that the theodolite must be accurately centred over the station point by means of a plumb-bob. It must also be carefully levelled, using the large level attached to the telescope or vertical arc, and not the small levels on the upper plate. The levelling should be tested and corrected before and after each round.

The stand must be firmly planted on the ground. If this be at all soft, stakes about 4 inches square should be driven down firmly, and cut off with a notch in each, to receive the point of the legs.

The effects of errors of adjustment in the theodolite, and their elimination, have been fully discussed under accurate traverses in Part I.

Error due to Displacement of Signal.—Error due to displacement of the signal is probably the *greatest* and *most common* source

of error. When possible the observation should be made to the *point* of a staff held on the station point, or at least *as low down as possible.* Unfortunately this cannot always be done, since observations have often to be made to the summit of a signal, perhaps 15 to 20 feet high. This error cannot be eliminated by "reiteration" or "repetition." There is nothing for it, therefore, but to take the utmost care in setting up and securing the signals, so that the summit observed is, and remains, perpendicularly above the station point. If the signal is 30 feet high or more, it will be best to use the theodolite to set it up, as follows. Place the foot of the pole, when a tripod is not used, a little to one side of the station mark, and then set up and adjust the theodolite some 50 feet or more from the station mark. Bisect the station mark, and clamp the limbs, elevate the telescope, and by means of guy-ropes make the apex of the pole coincide with the cross-wires. Make fast the guys. Repeat this operation in a plane at right angles to the first, using two other guys. The summit is now perpendicularly above the station point.

Error of Bisection.—Error of bisection, *i.e.* the cross-hairs are not directed exactly to the centre of the signal. With care and with the short sides used in minor triangulation, this error should not be serious. It depends upon several factors, viz. the accuracy of focussing, the eye of the observer, the defining-power of the instrument, the definiteness of the signal, and the state of the atmosphere.

Most of these have already been referred to. As regards the state of the atmosphere, in hot climates the latter often "boils" at certain times of the day. When this takes place it is useless to observe, the signal will dance about and become distorted in so erratic a manner that accurate work is impossible. To attempt to observe under these conditions merely amounts to loss of time and temper.

Testing of Instruments at Kew Observatory.—Theodolites and sextants can be tested at the Kew observatory for a very moderate fee, in all points, such as graduation, quality of telescope, etc. When ordering from abroad, it is well to ask for a Kew certificate. This should never be omitted in the case of sextants, which cannot be tested by the surveyor, and of which the graduation errors are often much greater than would be supposed. (See also p. 11.)

Angles from Magnetic North.—As most theodolites are provided with a compass, some surveyors prefer to set to zero on magnetic north for the first round when reading the angles, instead of setting on one station. This gives a rough idea without calculation of the actual direction of each line.

Form of Field Book and Entries therein.—The annexed (Table C) is a convenient form of "Field Book" for recording observation, and refers to Fig. 28, p. 84.

The "mean bearings" having been worked out in the Field Book, the included angles from station to station are next computed by successive subtraction, deducting each "bearing" from the next greater. The observed value of the several angles of each triangle should then be inscribed each in its position on the "diagram of triangulation" (made during the location of the points), or on a rough diagram made by plotting the angles with a common protractor.

Selection of Size of Theodolite.—The size of theodolite to be employed in a minor triangulation (setting aside money considerations), depends a good deal on the nature of the country to be surveyed. In any case an 8-inch theodolite of the transit pattern would suffice for every purpose. With such an instrument "azimuths" and "latitudes" could be obtained with a considerable degree of accuracy. If a telegraph station were at hand, whereby time signals could be obtained from some standard timepiece, "longitudes" could also be obtained with some degree of precision.

An 8-inch theodolite in open country, where transport is easy, is very suitable. In rough country a smaller and lighter instrument would be preferable. The larger and therefore the heavier the instrument, the more liable it is to injury in transport, and the greater is the cost and delay in carrying it from place to place and in setting it up. Generally, and especially in rugged country, the lightest instrument that will give the desired results should be used. These can be obtained (in a triangulation having one-mile sides) by means of a 5-inch theodolite. In a triangulation of this description which the writer once inaugurated and for some time directed, an old-fashioned 5-inch cradle theodolite was used by persons who had little previous experience of the work. The means available for erecting signals and the like were limited, yet the work was so far satisfactory that in adjusting the polygons a maximum correction of 25 seconds was rarely needed. As the surveyors obtained more experience and skill, still better results were obtained.

With the perfect graduation which is now attainable, precision depends rather on design and workmanship than on size. In the great triangulation connecting Natal with the Cape Colony, better results were obtained with a 10-inch theodolite of special construction than with an 18-inch of more ordinary construction. Moreover, the results attained by means of these relatively small instru-

TABLE C.

HORIZONTAL ANGLES OBSERVED AT STATION N.

Date. *Observer.*

Station observed to.	Face left.			Face right.			Mean bearing.	Included angle.
	Vernier A.	Vernier B.	Vernier C.	Vernier A.	Vernier B.	Vernier C.		
	° ′ ″	′ ″	′ ″	° ′ ″	′ ″	′ ″	° ′ ″	° ′ ″
A	81 38 19	38 21	38 26	261 38 19	38 20	38 15	81 38 23	42 26 30
M	124 4 51	5 2	4 52	304 4 49	4 42	5 2	124 4 53	79 30 30
L	203 35 14	35 29	35 27	23 35 31	35 19	35 18	203 35 23	80 55 32
O	284 30 43	30 50	31 10	104 30 49	30 59	30 59	284 30 55	51 18 13
R	335 49 2	48 58	49 21	155 49 8	49 13	49 6	335 49 8	50 23 45
B	26 12 57	13 6	12 42	206 12 46	12 54	12 53	26 12 53	55 25 30
A	81 38 21	38 23	38 28	261 38 30	38 29	38 25	—	

ments were superior to those obtained with the 3-foot circle in the earlier days of the Indian Survey.

Five-inch theodolites are now made by Troughton & Simms, and doubtless by others, in which micrometer microscopes are substituted for verniers. By this means readings to 10 seconds and less can be obtained with ease. Setting aside the question of accuracy, it is the universal opinion that the angles can be read by micrometers more quickly, easily, and pleasantly, than by means of verniers, and that there is less liability to error. Such an instrument would doubtless be most useful for all survey work.

Methods of filling in Details.—The method to be used in filling in the detail of a triangulation depends upon the scale of the map or plan which is to be produced. In the case of the Topographic Map of India, published on the scale of 1 inch to 4 miles, a plane-table of the simplest character was used. Traversing was not resorted to except in the case of important boundaries. The plane-table work was surveyed in the field on a larger scale and subsequently reduced by photography.

If a map on the scale of $\frac{1}{60000}$ (a little larger than 1-inch to 1 mile) has to be produced, the plane-table, compass and chain, or even compass and pacing may be used. In this case also it would be well to plot and fill in to a larger scale, say $\frac{1}{10000}$, and reduce afterwards by photography. With so small a scale as $\frac{1}{60000}$ many features such as roads and paths have to be conventionalized.

These simple methods will not suffice in the case of a plan on the scale of $\frac{1}{2500}$, or even of $\frac{1}{10000}$. The English Ordnance Survey practice is to proceed at once to chain surveying, measuring the sides of triangles, taking up detail on the way, and subdividing each main triangle by as many more lines as may be necessary to delineate detail. Traversing is avoided to the utmost, why, the writer is at a loss to understand, unless, indeed, it be owing to the fact that at the beginning of the century, when the Ordnance Survey was inaugurated, computation of traverses and plotting them by co-ordinates was not generally known. The uncertain protractor was then the only means available.

As a fact, traverse surveying may easily be conducted with greater accuracy than general chain surveying, for the simple reason that the line along which the measurements are made is usually a road or path, which is favourable to accuracy, and there is every possible check.

In the survey of Malta, to which reference has been made (the scale being $\frac{1}{2500}$), the following procedure was followed. The roads, lanes, and paths were traversed, the measurements being made with steel bands. Every detail within the range of a tape was accurately taken by offsets set

out with an optical square. All these measurements were made with a degree of accuracy which fitted them for the production of a plan on the scale of $\frac{1}{500}$, the scale that was used for detail plans of villages.

The traverses were closed on and adjusted to the trigonometrical points. The interior details of fields and gardens, which were excessively complicated, were then filled in by means of a plane-table.

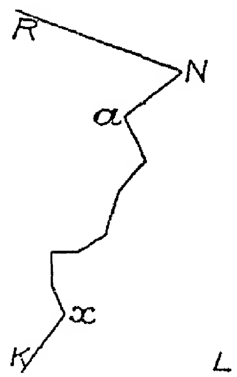


FIG. 40.

The method of adjusting the co-ordinates of a traverse to the triangulation is similar to that used when a subsidiary traverse branches off a main one, as in the survey on p. 178, Part I.

Start at a triangulation station, say N (Fig. 40), and measure the angle between any triangulation line, say NR, and the first traverse line Na. Then proceed as usual, measuring the included angles clockwise from the back station and chaining lengths. If we are working towards station K in the triangulation, we read and measure K from the last traverse station, x, then set up at K and read the angle to some other triangulation station, say x KL.

(a) To check the angles.

Obtain the bearing of the first back line, RN, from the computation of co-ordinates sheet, p. 82. If the bearing of RN is not given directly, but the whole circle bearing of say RS is given, we must then find from the diagram of triangulation (p. 84) the clockwise angle SRN and add.

$$\begin{array}{rrrr} \text{Thus suppose W.C.B. of RS} = & 25^\circ & 38' & 31'' \\ \text{Angle SRN} = 53^\circ\ 11'\ 24'' + 76^\circ\ 59'\ 13'' = & 130 & 10 & 37 \\ \hline \therefore \text{ Bearing of RN} = & 155 & 49 & 8 \end{array}$$

Next find in the same way the bearing of the final reference line KL, and *subtract from it* the bearing of the first line RN, first adding 360°, if necessary, for this purpose.

The sum of the observed traverse angles (including those at N and K) should be greater than the above result by exactly $n \times 180^\circ$, where n is an integral number.

$$\begin{array}{rrrr} \text{Thus suppose in this case bearing of KL} = & 78^\circ & 20' & 56'' \\ & 360 & 0 & 0 \\ \hline & 438 & 20 & 56 \\ \text{Bearing of RN (as above)} & 155 & 49 & 8 \\ \hline \text{Difference} & 276 & 31 & 48 \\ \text{Now suppose the sum of the observed angles} = & 1896^\circ & 34' & 0'' \\ & 276 & 31 & 48 \\ \hline \text{Difference} = & 1620 & 2 & 12 \end{array}$$

This is greater than 9×180 by 2′ 12″, whence we infer that a total correction of 2′ 12″ must be *subtracted* from the observed angles to find the corrected angles. The whole amount is distributed over the various angles as described in Part I., and then starting from the whole circle bearing of RN, we should check exactly at the end to the bearing of KL.

The differences of latitude and the departures are then calculated.

The difference between the sum of the southings and that of the northings should be the same as the difference between the north co-ordinates of N and K as given in the triangulation sheet and any discrepancy not exceeding the allowable error is distributed over the various lines before calculating co-ordinates.

If the traverse is plotted graphically it is adjusted by the graphical process described in Part I., p. 208.

If tacheometry is used for details the stations are treated as for a traverse, but if the details are filled in entirely by the plane-table or by chain surveying, the chief use of the triangulation is to enable a fresh start to be made at each triangulation station, thus avoiding accumulating errors.

Chain surveying alone would probably be seldom used nowadays.

The details are plotted as usual, according to the method adopted.

Examination of Completed Surveys in the Field.—Lastly, the finished sheet should be examined and corrected in the field. For this purpose a plane-table of simple construction would be found useful.

No important survey should be published until it has been so verified. This process, though important, is simple enough after a little practice. The survey checker is provided with a finished sheet, or a tracing thereof, a simple plane-table (mainly to act as a drawing-board), a tape, scale, and dividers. Going to some commanding point, he examines the country round to see that no fences or buildings have been omitted. He then perambulates the roads, seeing that no buildings or bends in the fences have been omitted.

Levels of Triangulation Stations.—In some cases it may be desired to find the relative levels of the triangulation stations by observing the vertical angles at the same time as the horizontal. For this, the curved shape of the Earth must be taken into account.

Let A and B (Fig. 41) be two points of different elevation, OA the vertical at A, and AC perpendicular thereto, so that AC is horizontal at A.

Let the angle of elevation, CAB to B, be observed at A, and let AD be a "level surface" through A (*vide* Part I.).

Then the difference of level BD is made up of two parts added together, namely DC and CB. Now, setting aside refraction for the present, CD is the correction for curvature, which can be computed with practical accuracy by the formula on p. 156, Part I. Rigorously, the distance BC is obtained from the triangle ABC as follows:—

Put AOB = angle subtended at Earth's centre by AB = θ;

$$\text{CAB} = \alpha, \text{ and AO} = r_1$$
$$\text{Then ABC} = 180^\circ - \text{BAO} - \theta$$
$$= 180^\circ - (90^\circ + \alpha) -$$
$$= 90^\circ - (\alpha + \theta)$$
$$\text{Also AC} = r_1 \tan\theta$$
$$\text{Now } \frac{\text{CB}}{\text{AC}} = \frac{\sin\alpha}{\sin \text{ABC}} = \frac{\sin\alpha}{\sin\{90^\circ - (\alpha+\theta)\}} - \frac{\sin\alpha}{\cos(\alpha+\theta)}$$
$$\therefore \text{CB} = r_1 \tan\theta \,.\, \frac{\sin\alpha}{\cos(\alpha+\theta)}$$

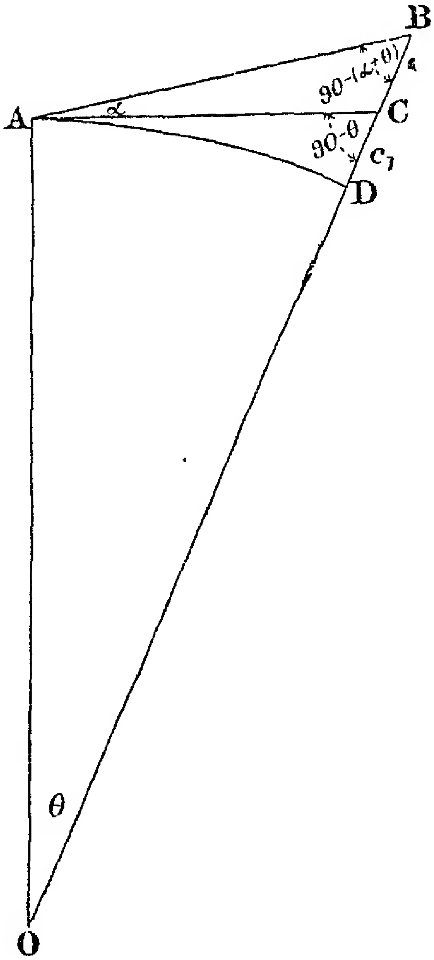


FIG. 41.

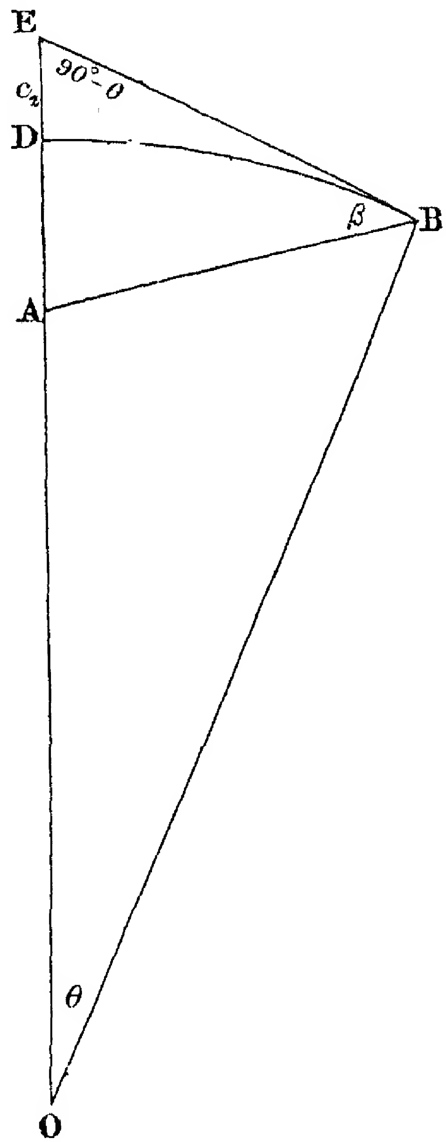


FIG. 42.

Now, in a minor triangulation the value of θ would rarely, if

ever, be greater than 10 minutes of arc, this corresponding with a line about 12 miles long.

We may therefore simplify the above formula.

It is evident that $\cos(\alpha + \theta)$ will be very nearly equal to $\cos \alpha$.

$$\text{Hence } \frac{\sin \alpha}{\cos(\alpha + \theta)} = \tan \alpha, \text{ very nearly}$$

Even in the extreme case where $\theta = 10'$ and $\alpha =$ say 5° (corresponding with a difference of level of about 1 mile in a distance of 12 miles), we have—

$$\frac{\sin \alpha}{\cos(\alpha + \theta)} = 0{\cdot}087511$$
$$\tan \alpha = 0{\cdot}087489$$

a difference of about 1 part in 4000 only, diminishing rapidly for less extreme cases.

Again, $r_1 \tan \theta$ may be taken as equal to the horizontal distance between A and B (as found from the triangulation).

Hence CB = *distance* × *tan angle of elevation.*

Let DC = c_1, the curvature and refraction correction, found as described on p. 156, Part I.

$$\begin{aligned} \text{Then rise A to B} &= \text{DB} = \text{DC} + \text{CB} \\ &= \text{distance} \times \tan \alpha + c_1 \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

If BE (Fig. 42) be the horizon at B (at right angles to BO) and the angle of depression β be measured to A, then AD is the true difference of level, and DE = c_2 is the curvature correction.

$$\begin{aligned} \text{Now EAB} &= \text{AOB} + \text{OBA} \\ &= \theta + 90^\circ - \beta \\ &= 90^\circ - (\beta - \theta) \end{aligned}$$

$$\begin{aligned} \therefore \text{AE} &= \text{BE} \frac{\sin \beta}{\sin\{90^\circ - (\beta - \theta)\}} \\ &= r_2 \tan \theta \times \frac{\sin \beta}{\cos(\beta - \theta)}, \text{ where BO} = r_2 \end{aligned}$$

As before, we may write this—

$$\begin{aligned} \text{AE} &= \text{distance} \times \tan \beta, \text{ very nearly} \\ \therefore \text{fall B to A} &= \text{AE} - \text{DE} \\ &= \text{distance} \times \tan \beta - c_2 \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

To allow for refraction, c_2 is simply made equal to the combined curvature and refraction corrections.

Reciprocal Angles.—If we wish to find the difference of level between A and B from the *mean* of the results, we must add the rise A to B and the fall B to A together and take the mean.

Put D = distance between stations.

Rise $= \mathrm{D} \tan \alpha + c_1$.

Fall $= \mathrm{D} \tan \beta - c_2$.

Mean $= \mathrm{D} \times \frac{1}{2}(\tan \alpha + \tan \beta) + c_1 - c_2$.

Now, so far as curvature goes, c_1 and c_2 are practically identical in minor triangulation. For a difference of level of 1 mile, the difference is only about 1 part in 4000, and as the curvature correction for an extreme distance of 12 miles is only about 100 feet, one part in 4000 means only about 0·3 inch, a negligible amount.

And if the observations are made under the same atmospheric conditions (that is, as far as possible at the same moment), the refraction correction may be regarded as the same also.

Hence we may put $c_1 = c_2$ in this case, and the formula becomes—

$$\text{rise} = \mathrm{D} \times \tfrac{1}{2}(\tan \alpha + \tan \beta) \quad . \quad . \quad . \quad . \quad (3)$$

Lastly, if α and β are both small, we may put $\frac{1}{2}(\tan \alpha + \tan \beta) = \tan \frac{1}{2}(\alpha + \beta)$ nearly.

Even for a distance of 12 miles α and β should not differ by more than about 10 minutes of arc, as will be shown later.

And taking $\alpha = 5^\circ$ (an extreme figure for this distance) and $\beta = 5^\circ\ 10'$, we have—

$$\tfrac{1}{2}(\tan \alpha + \tan \beta) = 0{\cdot}0889546$$
$$\tan \tfrac{1}{2}(\alpha + \beta) = 0{\cdot}0889544$$

again a negligible difference. Hence we may write finally—

$$\text{rise} = \text{distance} \times \tan \tfrac{1}{2}(\alpha + \beta) \quad . \quad . \quad . \quad . \quad (4)$$

This formula is perfectly general for minor triangulation. It gives the *rise* from A to B, where α is the angle of *elevation* from A to B and β is the angle of *depression* from B to A.

If the angles are *depression* from A to B and *elevation* from B to A, then both α and β will be negative in the formula. The result will also be negative, indicating a *fall* from A to B.

Finally, as will be shown later, *both* angles may be angles of depression.

In this case α will have the *minus* sign in the formula, while β will be positive, and the result will be positive or negative according to whether β is numerically greater or less than α, indicating accordingly a *rise* or *fall* from A to B.

Eye and Object Correction.—Before giving numerical examples,

we must consider the modifications in these formulæ which become necessary in consequence of the fact that the height of the theodolite (or eye) above the ground is not usually the same as the height of the signal or object to which the angle is read.

If the theodolites at A and B were the same height above the ground, and each of the signals observed was also of this same height, the angles would be truly reciprocal, as we have hitherto assumed (remembering also, however, that they must be observed almost simultaneously and under suitable conditions, to give equal refraction).

If not, each of the formulæ (1) and (2) above gives the difference of level from the centre of the telescope to the point observed on the signal.

The correction for this is called the "eye and object" correction, and for minor triangulation it is found in exactly the same way as in tacheometry.

That is, in working from a station A to another station B, we must call the height of the theodolite at A *plus* and the height of the signal at B *minus*. Then the algebraical sum of these is to be applied algebraically to the calculated *rise* from A to B, a calculated *fall* being called *minus*. The final result if *plus* means a *rise*, and if *minus* a fall. Hence, when reading vertical angles, it is always necessary to measure the height of the instrument above the ground, and to note the height of the point observed above the ground.

Of course, if the observations are made simultaneously the theodolites can be set up at about the same height, and this correction is unnecessary.

But frequently a greater height of signal than this is necessary to be visible at all, and the observations must simply be taken as nearly as possible under the same conditions.

Put T_A = height of theodolite at A.
S_A = height of signal at A.
T_B = height of theodolite at B.
S_B = height of signal at B.

Then formula No. 1, p. 103, becomes—

$$\textit{Rise}\text{ A to B} = \text{distance} \times \tan \alpha + c_1 + T_A - S_B \quad . \quad (5)$$

Here α is the angle of elevation A to B, and c_1 is the curvature and refraction correction.

Similarly if β be the angle of depression from B to A, we have—

$$\textit{Rise}\text{ B to A} = -\text{ distance} \times \tan \beta + c_2 + T_B - S_A$$

If this sum be negative it shows a *fall* from B to A; or we can change all the signs on the right, and so obtain the formula for the fall directly. Thus—

$$\text{Fall B to A} = \textit{distance} \times \tan\beta - c_2 - T_B + S_A \quad . \quad (6)$$

Hence to find the rise A to B on the mean of the results we take the mean of the rise A to B and fall B to A, and put $c_2 = c_1$ and $\frac{1}{2}(\tan\alpha + \tan\beta) = \tan\frac{1}{2}(\alpha + \beta)$ as before.

Hence mean value of *rise* A to B

$$= \text{distance} \times \tan\tfrac{1}{2}(\alpha+\beta) + \tfrac{1}{2}\{T_A + S_A - (T_B + S_B)\} \quad (7)$$

This, as before, is a perfectly general algebraic formula; A is the *back* station; T_A and S_A the heights of theodolite and signal there. These are both *plus*. Theodolite and signal at the *forward* station, B, are both *minus*; and the eye and object correction is *one-half* the algebraic sum of these.

Numerical Example.—For example, take the following observations:—

Station.	Point observed.	Angle of elevation.	Heights +		Line.	Log of distance.
			Theodolite.	Signal.		
		° ′ ″				
A	B	+2 43 4	4·32	13·92	AB	3·7244888
B	A	−2 33 32	5·11	11·36	BA	
B	C	−0 1 0	4·18	5·73	BC	4·0990934
C	B	−0 0 15	4·75	5·11	CB	

The reduced level of A is 275·36, and all linear dimensions are in feet. Required the levels of B and C.

The data are ideal and devoid of error. The object is not to show the degree of accuracy attainable, but merely to show the methods of calculation.

We shall first calculate the results by formula No. 7.

$$\begin{array}{rl}
\text{For the line AB, } \alpha = & 2^\circ\ 43'\ \ 4'' \\
\beta = & 2\ \ \ 33\ \ 32 \\
\hline
\dfrac{\alpha+\beta}{2} = & 2\ \ \ 38\ \ 18 \\
\log\tan = & 8{\cdot}6635141 \\
\log\text{distance} = & 3{\cdot}7244888 \\
\hline
& 2{\cdot}3880029
\end{array}$$

$$\text{Antilog} = +244{\cdot}35$$

$$\begin{array}{lll} T_A = 4{\cdot}32, & S_A = 11{\cdot}36, & \text{sum} = 15{\cdot}68 \\ T_B = 5{\cdot}11, & S_B = 13{\cdot}92, & \quad ,, \ = 19{\cdot}03 \end{array}$$

$$\begin{aligned} \text{Difference} &= -3{\cdot}35 \\ \tfrac{1}{2} &= -1{\cdot}67 \\ \therefore \text{ rise A to B} = +244{\cdot}35 - 1{\cdot}67 &= 242{\cdot}68 \\ \text{Level of A} &= 275{\cdot}36 \\ \hline \therefore \quad ,, \quad \text{B} &= 518{\cdot}04 \text{ feet} \end{aligned}$$

$$\begin{aligned} \text{For line BC, } \alpha &= -0^\circ \ \ 1' \ \ 0'' \\ \beta &= +0 \ \ \ 0 \ \ 15 \\ \hline \tfrac{1}{2}(\alpha + \beta) &= -0 \ \ \ 0 \ \ 22{\cdot}5 \end{aligned}$$

$$\begin{aligned} \text{log tan} &= 6{\cdot}037757 \\ \text{log dist.} &= 4{\cdot}099093 \\ \hline & 0{\cdot}136850 \\ \text{antilog.} &= -1{\cdot}37 \end{aligned}$$

$$\begin{array}{lll} T_B = 4{\cdot}18, & S_B = 5{\cdot}11, & \text{Sum} = 9{\cdot}29 \\ T_C = 4{\cdot}75, & S_C = 5\ 73, & \quad ,, \ = 10{\cdot}48 \end{array}$$

$$\begin{aligned} \text{Difference} &= -1{\cdot}19 \\ \tfrac{1}{2} &= -0{\cdot}60 \\ \therefore \text{ rise B to C} = -1{\cdot}37 - 0{\cdot}60 &= -1{\cdot}97 \\ \text{Level of B} &= 518\ 04 \\ & -1{\cdot}97 \\ \hline \text{Level of C} &= 516{\cdot}07 \text{ feet} \end{aligned}$$

We shall now, for example, calculate the lines A to B and B to C, using the forward angles only.

We shall calculate the curvature correction as $\dfrac{(\text{distance})^2}{\text{diameter of earth}}$ (p. 156, Part I.), and the refraction correction as *one-seventh* of this, and work by formula No. 5, p. 105.

$$\begin{aligned} \text{For the line AB log distance} &= 3{\cdot}7244888 \\ \text{Double for square} &= 7{\cdot}4489776 \\ \text{log diameter of earth in feet, say } & \ 7{\cdot}6209720 \\ \hline & \ \bar{1}{\cdot}8280056 \end{aligned}$$

$$\begin{aligned} \text{Curvature correction} = \text{antilog} &= 0{\cdot}67 \text{ foot} \\ \text{Refraction} &= 0{\cdot}10 \ \ ,, \\ \hline \text{Curvature and refraction} = c_1 &= 0{\cdot}57 \ \ ,, \end{aligned}$$

$$\begin{aligned} \text{For the line BC, similarly, curvature} &= 3{\cdot}78 \\ \therefore \text{ refraction} &= 0{\cdot}54 \\ \hline \therefore c_1 &= 3{\cdot}24 \end{aligned}$$

This is always to be reckoned *plus*.

$$\begin{aligned} \text{For the line AB, } \alpha &= 2^\circ\ 43'\ 4'' \\ \log \tan &= 8 \cdot 676417 \\ \log \text{distance} &= 3 \cdot 724489 \\ \hline & \quad 2 \cdot 400906 \end{aligned}$$

$$\begin{aligned} \text{antilog} &= +251 \cdot 71 \\ T_A = 4 \cdot 32, \quad S_B &= 13 \cdot 92 \\ \therefore \text{rise A to B} &= +251 \cdot 71 + 0 \cdot 57 + 4 \cdot 32 - 13 \cdot 92 \\ &= 242 \cdot 68 \text{ feet} \end{aligned}$$

This agrees exactly with the result previously obtained, because the dates were chosen to make it do so. Such agreement could not be expected in practice, and of course the result from the *mean* of the angles is to be preferred.

$$\begin{aligned} \text{For the line B to C, } \alpha &= -0^\circ\ 1'\ 0'' \\ \log \tan &= 6 \cdot 463726 \\ \log \text{distance} &= 4 \cdot 099093 \\ \hline & \quad 0 \cdot 562819 \end{aligned}$$

$$\begin{aligned} \text{antilog} &= -3 \cdot 65 \\ T_B = 4 \cdot 18, \quad S_C = 5 \cdot 73, \quad c_1 &= 3 \cdot 24, \text{ as above} \\ \therefore \text{rise B to C} &= -3 \cdot 65 + 3 \cdot 24 + 4 \cdot 18 - 5 \cdot 73 \\ &= -1 \cdot 96 \text{ feet} \end{aligned}$$

which is also practically the same as before.

Arrangement in Circuits.—Usually the rises and falls are calculated in circuits working from one station back to the same station; the sum of the rises in such a circuit should be equal to the sum of the falls, and any small discrepancy is distributed over the different lines, in the same way as when calculating co-ordinates for a closed traverse.

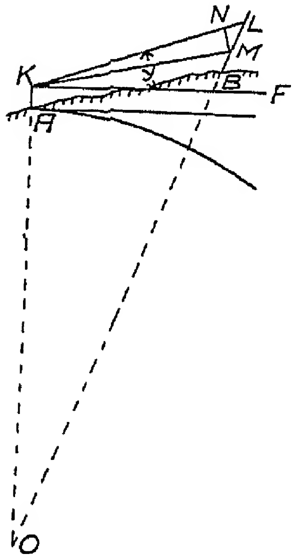


Fig. 43.

The correction to any line should be proportional to the length of line, *not* to the rise or fall.

Rigorous Computation. — The above formulæ, though amply sufficient for minor triangulation, are, of course, not quite rigorous.

In working more rigorously, the chief differences are: (*a*) that in making the eye and object correction, we calculate the effect on the vertical angle due to the difference of heights, and apply this as a correction to the observed angle, so as to make the angles more truly reciprocal; (*b*) that we use a formula for the whole difference of level, including the curvature correction, so as to remove any error due to the variation in the latter.

Eye and Object.—To compute the change in the angle, let K (Fig. 43) be the centre of the theodolite at station A, and L the point observed on the signal at B, while M is a point at the same height above B as K is above A.

The angle of elevation is $\gamma = \text{FKL}$.

It is required to find the value of the reduction MKL due to the difference ML in the heights.

Draw MN perpendicular to KL.

Then, without sensible error in calculating a small correction, we may suppose KF perpendicular to BL.

$$\therefore \text{KLB} = 90^\circ - \gamma, \quad \therefore \text{NML} = \gamma$$

$$\text{hence NM} = \text{ML} \cos \gamma \text{ very nearly}$$

$$\text{But } \tan \text{MKL} = \frac{\text{NM}}{\text{KN}}$$

$$= \frac{\text{ML} \cos \gamma}{\text{KL}} \text{ very nearly}$$

Now $\text{KL} = \text{KB} \sec \gamma$, and we have seen that KB is practically equal to the horizontal distance, whence we have—

$$\tan \text{MKL} = \frac{\text{ML} \cos \gamma}{\text{KB} \sec \gamma}$$

$$= \frac{\text{diff. of heights} \times \cos^2 \gamma}{\text{horizontal distance}}$$

Such assumptions as we have made will cause merely a very small percentage error in the result, which will be a negligible quantity as the whole result is itself small.

Moreover, for small angles,

$$\tan \text{MKL} = \tan 1'' \times \text{angle MKL in seconds}$$

$$\text{MKL in seconds} = \frac{\text{diff. of heights} \times \cos^2 \gamma}{\text{horizontal distance} \times \tan 1''}$$

The "difference of heights" here means the result of *subtracting the height of theodolite at A from the height of signal* BL *at* B. The result may be plus or minus, and is to be *subtracted* algebraically from angles of *elevation*, and *added* to angles of depression, as observed at A.

Omission of $\cos^2 \gamma$.—Suppose we place some limit, say 10″, as the greatest degree of accuracy at which we can hope to aim in the vertical angles. Then in many cases we may omit the factor $\cos^2 \gamma$; say whenever, by doing so, we do not affect the correction by as much as 5″.

This depends upon (*a*) the ratio $\frac{\text{ML}}{\text{KB}} = \frac{h}{d}$, say.

(*b*) the value of γ.

The annexed table gives a series of values of $\frac{h}{d}$ with the corresponding eye and object corrections, calculated *without* considering the $\cos^2 \gamma$.

Below these is given the error in seconds arising from the neglect of this factor.

h/d.	0·001	0·002	0·003	0·004	0·005	0·006	0·007	0·008	0·009	0·01
Corrn. in secs.	206·3	412·5	618·8	825·0	1031	1238	1444	1650	1856	2063
Vertical angle.	Error in seconds									
1°	0·06	0·13	0·19	0·25	0·31	0·38	0·44	0·50	0·57	0·63
2°	0·25	0·50	0·75	1·00	1·26	1·51	1·76	2·01	2·26	2·51
3°	0·56	1·13	1·69	2·26	2·82	3·39	3 95	4 52	5·08	5·65
4°	1·00	2·01	3·01	4·01	5·02	6·02	7·03	8·03	9·03	10·04
5°	1·57	3·13	4·70	6·27	7·83	9·40	10·97	12·54	14·10	15·67
7½°	3·51	7·03	10·54	14·06	17·57	21·08	24·60	28·11	31·63	35·14
10°	6·22	12·44	18·66	24·88	31·10	37·32	43·54	49·76	55·98	62·20
12½°	9·66	19·33	28·99	38·65	48·31	57·98	67·64	77·30	86·96	96·62

For greater ratios of $\frac{n}{d}$, the values for any angle are proportional. Thus if $\frac{h}{d} = 0{\cdot}05$, $\gamma = 3°$, we find that for 0·005 with $\gamma = 3°$ the error is 2·82; hence for 0·05 we must multiply by 10, giving 28·2 secs. The table thus shows how far this factor may be neglected. Reduced to an empirical formula, we see from the table that we may omit $\cos^2 \gamma$ if γ be less than $\sqrt{0{\cdot}08 \div \frac{h}{d}}$, and the error will not exceed 5″. γ here is in degrees.

Formula for Rise.—The angles, after these corrections are applied, must be regarded as reciprocal.

In Fig. 44, let α and β be the reciprocal angles, as obtained above (still assuming no refraction) at A and B, the former elevation, the latter depression.

We are supposed to be working from A to B, and AD is the horizontal distance AB at the level of A.

It is required to find the rise DB.

We assume, for the moment, that there is no refraction.

Then $\quad OBA = 90° - \beta$

$\quad\quad BAO = 90° + \alpha$

and let $\quad AOB = \theta$

$$\therefore\ OBA + BAO + AOB = 180° - \beta + \alpha + \theta$$

But the sum of these angles $= 180°$

$$\therefore\ \beta = \alpha + \theta$$

That is, *the angle of depression from* B *to* A *is greater than the angle of elevation* A *to* B, *by the angle* θ *which* AB *subtends at the Earth's centre.*

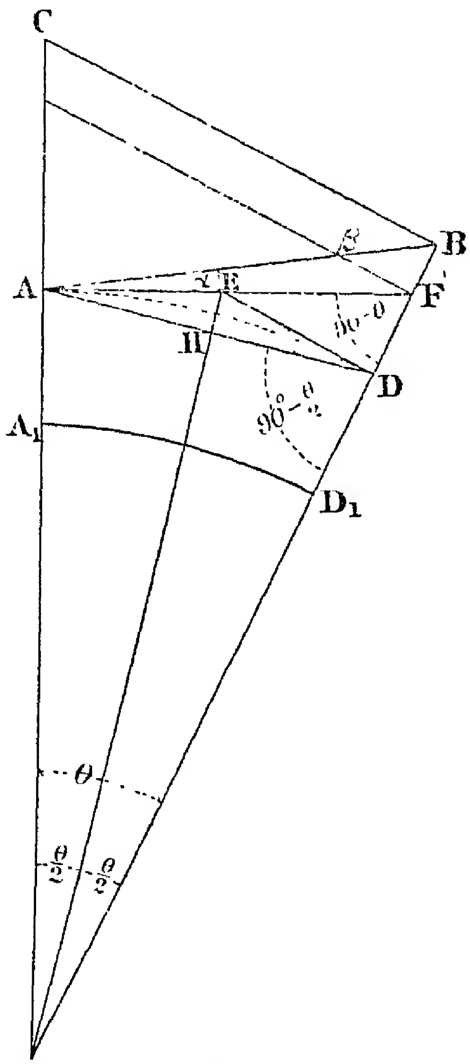


FIG. 44.

Now AF is a tangent at A to the arc AD ; hence

$$\text{FAD} = \tfrac{1}{2}\text{AOD} = \frac{\theta}{2} = \frac{\beta - \alpha}{2}$$

$$\therefore\ \text{DAB} = \alpha + \text{FAD} = \frac{\beta + \alpha}{2}$$

Hence $$\frac{\text{DB}}{\text{AD}} = \frac{\sin\frac{1}{2}(\beta + \alpha)}{\sin(90^\circ - \beta)}$$

$$= \frac{\sin\frac{1}{2}(\beta + \alpha)}{\cos\beta}$$

and if r_1 be the radius of curvature corresponding to the arc AD, then $AD = 2r_1 \sin \frac{1}{2}\theta$, and $DB = 2r_1 \sin \frac{\theta}{2} \cdot \frac{\sin \frac{1}{2}(\beta + \alpha)}{\cos \beta}$.

The factor $2r_1 \sin \frac{\theta}{2}$ may in all practical cases be safely replaced by d, the horizontal distance. Even if d were 60 miles, the difference would scarcely be appreciable.

The ratio of the two factors for 60 miles is 1·000013, making, therefore, an error of 0·13 foot in the height of a mountain 10,000 feet high, corresponding with an error of less than one-tenth of a second of arc in the observed angle.

Both Angles Depression.—If both angles are angles of depression, we put β_2 = greater angle, β_1 = smaller. Then β_2 corre-

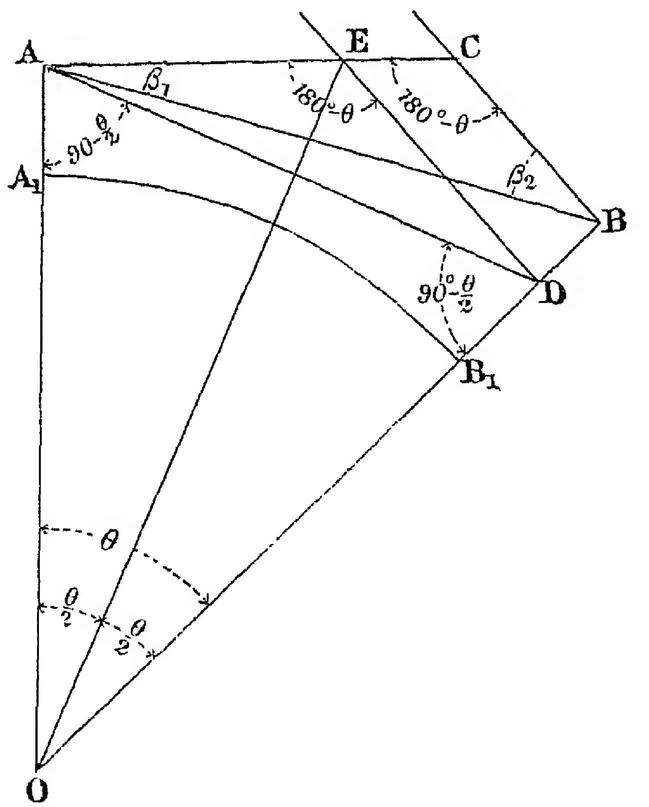


Fig. 45.

sponds to β in the above formula, and β_1 to α. But as the signs are opposite, we must write $-\beta_1$ instead of α.

The formula then becomes

$$2r_1 \sin \frac{\theta}{2} \cdot \frac{\sin \frac{1}{2}(\beta_2 - \beta_1)}{\cos \beta_2}$$

The student should have no difficulty in verifying this formula by reference to Fig. 45.

Formula for Horizontal Distance.—The sides as given by the triangulation will be, in general, the *geodetic distances* (see p. 80), corresponding with the arc A_1D_1 in Fig. 44 (or A_1B_1 in Fig. 45) at mean sea-level.

Now for $2r_1 \sin\frac{\theta}{2}$ we must substitute the *horizontal distance at the level of* A.

Let d = required horizontal distance AD.
g = known geodetic distance A_1D_1 (Fig. 44).
$A_1\bar{A} = h$ = height of A above mean sea-level.
$r = A_1O$ = radius of earth at M.S.L.

$$\text{Then } \frac{d}{g} = \frac{r+h}{r}$$

$$\therefore\ d = g\left(1 + \frac{h}{r}\right)$$

Hence the correction to be *added* is $g \times \frac{h}{r}$.

Usually, however, we are given the log distances, and require the log horizontal distances. We may find the correction to the logarithm directly as follows:—

$$\text{Log } d = \log g + \log(r+h) - \log r$$

Now $\quad \text{Log } r = \log 20{,}889{,}000 = 7{\cdot}3199176$

and $\quad \log 20{,}899{,}000 = 7{\cdot}3201255$

$$\text{Difference} = 0{\cdot}0002079$$

Also, between these two limits corresponding with an interval of 10,000 feet, we may obtain the logarithm of $r+h$ by proportion, as one interpolates in ordinary working.

$$\text{We may say } \log(r+h) = \log r + \frac{0{\cdot}0002079h}{10{,}000}$$

Putting this for $\log(r+h)$

$$\text{Log } d = \log g + \log r + \frac{(0{\cdot}0002079 \times h)}{10{,}000} - \log r$$

$$= \log g + \frac{(0{\cdot}0002079 \times h)}{10{,}000}$$

Example.—

$$h = 1{,}463{\cdot}3 \text{ feet } \log = 3{\cdot}1653$$
$$\log (2{\cdot}079 \div 10^8) = \bar{8}{\cdot}3178$$

Natural number of correction $0{\cdot}0000304 = \bar{5}{\cdot}4831$
Adding $\log g = 5{\cdot}1346620$ (say)

$$\log d = 5{\cdot}1346924$$

Effect of Refraction on Vertical Angles.—The effect of refraction has now to be considered. Rays of light passing through the atmosphere nearly parallel to the earth's surface do not travel in a straight line. Owing to the varying density of air, due to differences of pressure and to irregular heating, they travel in a curved line. Usually, almost invariably, the path

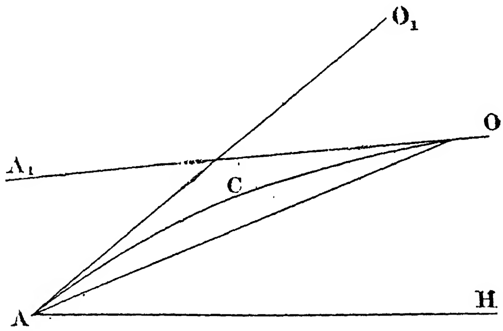


Fig. 46.

of the ray is *concave* towards the earth's surface, the effect of this being to make an object appear *higher* than it *really is.*

Thus if O be a signal and A the point of observation (Fig. 46) the rays of light would not proceed along the straight line OA but along a curved line OCA. The consequence is that the rays enter the eye of the observer at a greater angle than OAH. They enter it as if OCA were a tangent at A to that curve, and consequently the object O appears as though it were at O_1, and an erroneous angle O_1AH is observed. The angle O_1AO is the "refraction error." If now the eye were removed to O and directed to A, then OA_1 will also be tangent to the curve OCA and the apparent position of A would be at A_1. Assuming that the curve is symmetrical, then the angle A_1OA will be equal to O_1AO.

Each of these angles represents the refraction correction, which we shall call ρ.

$$\therefore\ A_1OA = O_1AO = \rho$$

As refraction makes an object appear *too high*, it is clear that it *increases* an angle of elevation and *decreases* an angle of depression.

Thus if β and α be the true angles of depression and elevation between two points, while D and A are the observed values of the same angles, then

$$\beta = D + \rho$$
$$\alpha = A - \rho$$

Now we have seen (p. 110) that $\beta - \alpha = \theta$, where θ is the subtended angle at the earth's centre of curvature.

$$\begin{aligned} \text{But } \beta - \alpha &= D + \rho - (A - \rho) \\ &= D - A + 2\rho \\ \therefore\ \theta &= D - A + 2\rho \end{aligned}$$

The angles D and A are understood to have been corrected for heights of eye and object.

They may be called the "observed reciprocal angles."

If we take the difference between these, therefore, having previously calculated θ from the known radius of the earth (see Chapter VI.) and the known horizontal distance, we shall *not* find that $D - A = \theta$, as we should if the true angles β and α were known, but $D - A$ will, in general, be *less* than θ.

The difference $\theta - (D - A)$ tells the value of ρ, for from the above equation we have

$$2\rho = \theta - (D - A)$$

On the average of many results, it has been found that $\theta - (D - A)$ is, as a mean value, rather less than *one-seventh* of θ, so that $\rho =$ about $\frac{\theta}{14}$ or $\frac{\theta}{15}$.

Refraction Correction in Levelling.—By this result we justify our rule that the refraction correction is *one-seventh* of the curvature in levelling (see p. 156, Part I.) Thus in Fig. 47, if $OAP = \rho$, OP is the refraction correction in level, and DF is the curvature correction.

Now we have seen that—

$$\text{DAF} = \frac{\theta}{2}$$

$$\text{and OAP} = \rho = \frac{\theta}{14} \text{ say}$$

$$\therefore \text{ OAP} = \frac{\text{DAF}}{7} \text{ about}$$

and if all the lines be practically horizontal, AO may be taken as equal in length to AF, and each of them at right angles to the vertical DO.

Hence OP will be about *one-seventh* of DF.

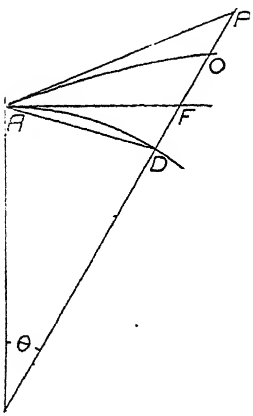


FIG. 47.

In Vertical Angles.—But for high angles it is clear that the above assumptions are no longer true, and the observed reciprocal angles should be further corrected by the quantity ρ before calculating heights.

The value of ρ is found from the formula $2\rho = \theta - (D - A)$. The finally corrected angles are then substituted in the formula on p. 112.

Both Angles Depression.—If both angles are depression, and D_1, D_2 are the "observed reciprocal angles" (*i.e.* after correction for eye and object) of which D_2 is the greater, then

$$2\rho = \theta - (D_2 + D_1)$$

This is derived from the previous formula by putting $-D_1$ for A, and D_2 for D, as before.

If the negative sign be always ascribed to the "refraction error" and to "depressions," and the positive sign to "elevations," the algebraic sum will be the "*corrected angle.*"

Thus, if $\rho = 10''$,

$$\begin{aligned} \text{and the observed angle of elevation A} &= +\ 2^\circ\ 10'\ 43'' \\ \rho &= -\ 0 \quad 0 \quad 10 \\ \hline \text{then the corrected angle } \alpha &= +\ 2 \quad 10 \quad 33 \end{aligned}$$

$$\begin{aligned} \text{If the observed angle of depression D} &= -\ 3^\circ\ 17'\ 06'' \\ \rho &= -\ 0 \quad 0 \quad 10 \\ \hline \text{the corrected angle } \beta &= -\ 3 \quad 17 \quad 16 \end{aligned}$$

It must not be forgotten that the correction thus obtained includes all errors of observation. Nor must it be supposed that the refraction correction is always the same.

The value of error of refraction, though usually small, is very uncertain. It varies at different hours of the day, being least (in the tropics) at about 3.45 p.m. It varies also with temperature and barometric pressure. Over highly heated arid plains it becomes exceedingly irregular, the curve sometimes being actually convex towards the earth, so that objects are *depressed* and not *raised*. The mirage is an extreme case of this. The rays of light proceeding from the sky become so much bent upwards on encountering the highly heated stratum of air immediately above the parched ground, that the observer looking in a nearly horizontal direction sees, not the ground and the objects thereon, but an image of the sky, and the appearance is that of a sheet of water. Horizontal deviation is also very common. A series of rods, ranged out in a straight line at one time, have presented the appearance of a curve at another time. When disturbance is so great as this, accurate observation of any kind is obviously impracticable. At any time, however, the effect of refraction is uncertain, and accurate results can only be arrived at by taking reciprocal angles under similar atmospheric conditions.

We may mention here a case in the Indian Survey (quoted in the article on "Geodesy" in the "Encyclopædia Britannica") where the vertical angle to a station $10\frac{1}{2}$ miles away varied from a depression of 4′ 53″ at 4.30 p.m. to an elevation of 2′ 4″ at 10.50 a.m.

"Terrestrial refraction" must not be confounded with "celestial or astronomical refraction," that is to say, the refraction which affects the observation of heavenly bodies. "Celestial refraction" at low angles is greater than "terrestrial," because the rays from the stars traverse the whole depth of the atmosphere, whilst rays from a terrestrial object pass through a limited section of it only, and in a comparatively horizontal direction.

Subtended Angle.—To find the angle θ when the geodetic distance is known, the radius of curvature proper to the place should be used. For further information on this point see Chapter VI.

But assuming a mean radius of 20,890,200 feet, the length of arc subtending an angle of *one second* is $\frac{2\pi r}{360 \times 60 \times 60}$, or 101·2 feet nearly. Hence

$$\theta = \frac{\text{distance in feet}}{101{\cdot}2} \text{ secs.}$$

The corresponding logarithm is 2·00552.

We must subtract this from log geodetic distance, and the result will give the log of θ expressed in seconds.

Numerical Example.—We shall now work the same example as before by the formulæ now explained.

For convenience the data are repeated.

Station.	Point observed.	Angle of elevation.	Heights.		Line.	Log distance.
			Theodolite.	Signal.		
		° ′ ″				
A	B	+ 2 43 4	4 32	13·92	AB	3·7244888
B	A	− 2 33 32	5·11	11·36	BA	
B	C	− 0 1 0	4·18	5·73	BC	4·0990934
C	B	− 0 0 15	4·75	5·11	CB	

(*a*) To find the subtended angle A to B.

$$\text{Formula: } \theta = \frac{\text{geodetic distance}}{\text{No. of feet in one sec.}}$$

Log geodetic distance	3·72449
Log feet in one second	2·00552
	1·71897

$$\therefore \theta = 52{\cdot}4''$$

(*b*) To find the log of "horizontal distance" AB.

$$\text{Formula: } \log d = \log g + h \times \frac{0{\cdot}0002079}{10{,}000} \quad \text{(p. 113)}$$

Log height of station A = log 275·36 feet . . .	2·4399
Add log 2·079 ÷ 10^8	$\bar{8}$·3178
Sum	$\bar{6}$·7577
The natural number corresponding to this logarithm is	0·0000057
Adding log "geodetic distance"	3·7244888
The sum is log "horizontal distance" . . .	3·7244945

(*c*) To find the eye and object correction to the observed angle at A.

$$\text{Formula: correction} = \frac{\text{diff. of heights} \times \cos^2 \gamma}{\text{horizontal distance} \times \tan 1''}$$

Height of signal at B	13·92 feet
Height of instrument at A	4·32 "
Difference . .	9·60 "

Log "horizontal distance"	3·72449
Log tan 1"	$\bar{6}$·68558
Sum	$\bar{2}$·41007
Arithmetical complement	1·58993
Log 9·6 feet (diff. signal and inst. height) . .	0·98227
Log cosine 2° 43'	9·99951
The same (to square value)	9·99951
Log "eye and object correction" for angle at A = log 373	2·57122

Correction = 6' 13"

Signal higher than instrument, *deduct* from angle of elevation.

Note.—The eye and object correction can be calculated still more rigorously, but the formula is more troublesome, and gives practically the same result as the above.

(*d*) "Eye and object correction," for observed angle at B.

Height of signal at A	11·36	
Height of instrument at B	5·11	
Difference . .	6·25	Log 0·79588
Log cos 2° 34' to nearest minute		9·99956
The same again (to square value)		9·99956
Arithmetical complement of "log distance," log tan 1" (from *c*)		1·58993
Sum log correction in seconds = log 243" . .		2·38493

Correction 4' 03"

To be *added* to angle of depression, as the signal is higher than instrument.

(*e*) To find the refraction correction.

Formula: $\rho = \frac{1}{2}\{\theta - (D - A)\}$ (p. 115)

	° ′ ″	° ′ ″
Observed depression at B	2 33 32	
Add "E and O correction," 243" (*d*) = 4' 03"	4 03	
Sum = corrected depression (D)		2 37 35
Observed altitude at A	2 43 04	
Deduct "E and O correction," 373" (*c*)	6 13	
Sum corrected altitude (A)		2 36 51
Difference = D − A		0 0 44
The subtended angle (θ), see (*a*)		0 0 52
The difference is twice the refraction		0 0 8
The correction for refraction is half the above, or		0 0 4

To be *added* to "depression" *deducted* from "elevation angles."

(*f*) Computation of difference of level B from A.

$$\text{Formula: horizontal distance} \times \frac{\sin \frac{1}{2}(\beta + \alpha)}{\cos \beta}$$

where $\beta = D + \rho$; $\alpha = A - \rho$ (p. 115).

	° ′ ″	° ′ ″
The apparent depression	2 37 35	
Add refraction	0 0 4	
Corrected depression $= \beta$		2 37 39
The apparent elevation	2 36 51	
Deduct refraction	0 0 4	2 36 47
Sum		2)5 14 26
Half sum = contained angle		2 37 13
Log sin contained angle = log sin 2° 37′ 13″ . .		8·6600733
Log secant depression = log secant 2° 37′ 39″ .		0·0004568
Log "horizontal distance"		3·7244945
Log "difference of level," 242·69 feet		2·3850246

(*g*) Calculation of level.

Level of station A	275·36 feet
Add rise A to B from (*f*)	242·69 „
Level of station B	518·05 „

This should be compared with the result on p. 107.

Hence it appears that the more simple method of tangents gives, in the cases which are likely to occur in the practice of "minor triangulation," results which differ inappreciably from those which are obtained by the rigorous but laborious process, and the errors are less than those of observation when an ordinary 5-inch or 6-inch theodolite is used.

The actual computation, indeed, is not more difficult by the rigorous methods. It is the numerous corrections that take time. If much work had to be done, tables might be prepared which would facilitate work; for instance, a table could be made giving by inspection the "subtended angle" in terms of the "distance."

A table on this principle is given in the Manual of Surveying for India. Nor would it be difficult to prepare a subtense-table, giving the "eye and object correction" by inspection. If such tables were to hand, the "rigorous method" would be as nearly quick as the "approximate."

The student should work out the results for the line BC for himself, and compare with the results given below.

(*a*) Subtended angle B to C $= \theta = 124''$.

(*b*) Log of "horizontal distance" of BC at level of B = 4·0991042.

(*c*) To find the "eye and object correction" for angle at B.
Here we may omit $\cos^2 \gamma$ (p. 110).
Eye and object correction = 26″.
(*d*) "Eye and object correction" for angle at C = 6″.
(*e*) Determination of refraction from B.

	°	′	″	°	′	″
Observed depressions at B	0	1	0			
Add "eye and object correction" . .	0	0	26			
The sum is "apparent depression"				0	1	26
Observed depression at C	0	0	15			
Add "eye and object correction" . .	0	0	6			
The sum is "apparent depression"				0	0	21
The sum of the two depressions is				0	1	47
But the subtended angle is				0	2	4
The difference is				0	0	17

Half of which is the correction for refraction to be added to both angles.

Say 8″ and 9″ respectively.

(*f*) Calculation of difference of level.

	°	′	″	°	′	″
The angle of depression at B	0	1	26			
Add refraction	0	0	8			
The sum is the corrected angle.				0	1	34
The angle of depression at C	0	0	21			
Add refraction.	0	0	9			
The sum is the corrected angle				0	0	30
The difference is				0	1	4
The half difference is the contained angle . .				0	0	32

Then log 32″	1·5051500
Log tan 1″	4·6855749
Log secant 0° 1′ 34″	10·0000000
Log horizontal distance	4·0991040
Difference of level 1·95 feet	0·2898289

Level of B	518·05
Fall from B to C	1·95
Level of C	516·10

Observation of Vertical Angles.—The "vertical angles" will be observed simultaneously with the "horizontal angles" of the triangulation. If the vertical limb be furnished with two verniers, both should be read. A reading "face right" and "face left"

should be taken, and the "mean vertical angles" deduced in the ordinary manner.

It is desirable that the level of the theodolite should be on the T piece, not on the telescope. The level is then corrected (after aiming the telescope at each signal) by means of the clip screws, and the cross hairs are then brought to the proper point on the signal by the tangent screw.

At each station, the "height of instrument" and "of signal" must be recorded. If the trigonometrical point be in a position difficult of access, it will be well to establish near to hand a bench-mark in some more convenient position, by using the theodolite as a level, or by observing an angle and measuring a distance.

Procedure when Reciprocal Angles are not obtained.—It may, however, happen that it is impossible to obtain reciprocal observations. For example, it may be required to determine the height of an inaccessible peak, the distance of which has been determined by intersection. In this case the only plan is to assume a value for the correction for refraction.

Value of Refraction as a Fraction of "Subtended Arc."—It is the general practice of goedetic writers to express this value as a fraction of the "subtended arc," that is, the angle subtended at the centre of the earth between the two stations, or the angle θ in Fig. 45. Usually, this fraction is found to be about $\frac{1}{15}$. Hence it is necessary to add or deduct this fraction of the contained arc to the observed angle of "depression" or "elevation" as the case may be.

Desirability, or the Reverse, of Vertical Angle Determination of Heights.—The question now arises as to the circumstances under which it is desirable, or the reverse, to determine heights by triangulation. The observation of vertical angles adds very materially to the time which would be occupied in observing horizontal angles alone. It does not, however, add materially to the total cost of triangulation, for the time actually spent in observing is small when compared with that spent in going from point to point, in locating points, and in putting up signals, work which has to be performed in any case. The computation of the differences of level is somewhat laborious, the result being that the levels of a number of points, distant from each other by from half a mile to a mile, are determined with approximate accuracy. The degree of accuracy will not (with the instruments likely to be used in connection with minor triangulation) be equal to that of good levelling, by which, moreover, a large number of intermediate points would be determined.

The question of the desirability of resorting to vertical angles

also depends upon the class of map or plan which it is desired to produce. If the triangulation be undertaken for the purpose of preparing a small-scale topographical map, and if the country be rugged, making levelling expensive, then vertical angles will be valuable, and the approximate levels of a number of points will be determined, which will be most useful as a basis for sketching in topographical detail.

If, on the other hand, it be desired to produce at once large-scale plans with the fullest detail; and if the country be open and level, so as to make levelling easy and cheap, then the observation of vertical angles is scarcely worth the trouble and cost. Levelling will give at least as accurate results, and the levels of numerous intermediate points may be determined without appreciable additional cost. The computation of levels is a matter of the utmost simplicity, and the system may be arranged so as to give complete checks on the work. Levelling, moreover, is an operation which may be deputed to persons of limited attainments. If levelling operations are deferred until the detail plan is finished, the surface of the ground may be covered with a number of spot-levels and bench-marks at a cheap rate. It is only necessary to give the leveller a tracing of the ground over which he is to extend his operations, and he can locate upon it in the field (often without any measurements, but in any case by most simple ones) the positions of the spot-levels which he has taken, indicating the position of the staff by means of a number on the plan, corresponding with the proper entry in the field-book.

As an example, the student may calculate the following circuit from an actual survey :—

Station.	Point observed.	Angle of elevation.	Heights.		Line.	Log geodetic distance.	Remarks.
			Theodolite.	Signal.			
		° ′ ″					
A	B	4 18 22	5·3	6·3	AB	3·24319	R.L. of A
A	C	− 4 24 52	5·3	0	AC	3·36011	= 466·7.
B	A	− 4 14 20	5·1	6·3			All linear
B	D	− 6 10 50	5·1	0	BD	3·51677	dimensions
C	D	− 2 49 30	4·8	2·3	CD	3·00326	in feet.
C	A	4 22 0	4·8	6·3			
D	B	5 55 45	5·5	0			
D	C	2 42 25	5·5	6·3			

Work in the order A, B, D, C, A. The reduced levels, after correction, are 466·7, 597·2, 247·7, 294·5, and 466·7. The example may be worked by either or both of the methods given.

Satellite Stations.—It may sometimes happen that a point cannot be found, at a spot where a station is desired, suitable for the erection of a theodolite, and mutually visible from all the adjacent stations without the erection of high signals or expensive clearing work.

Thus let A, B, D, E (Fig. 48), be four surrounding stations. About at C a station is desired, and there is a church spire there which is visible from A, B, D, and E, but unsuitable as a point at which to erect a theodolite.

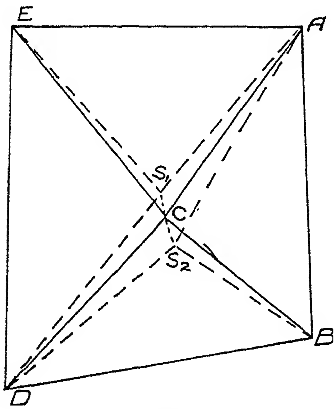


FIG. 48.

No other single point can be found from which we can easily see to A, B, D, and E, but it may be possible to find two points, S_1 and S_2, such that from the former we can see A, E, and D, and from the latter B, A, and D, while the spire C can also be seen from each of them.

Then C is chosen as the station, and the observations are taken to it from A, B, E, D.

Observations are taken from S_1 to A, E, D, and C, and from S_2 to B, A, D, and C, and it is required to deduce from these the values of the angles ACB, etc., as if they had actually been observed at C.

S_1 and S_2 are called satellite stations.

In addition to the angles at S_1 and S_2 the necessary data are—

(*a*) The distances AC, BC, etc.

These are obtained, sufficiently nearly, from the known stations A, B, etc., and the angles observed from them to C. The sides may be either calculated from these angles, or measured on a plan drawn by scale and protractor.

(*b*) The horizontal distances S_1C, S_2C.

These are measured directly, if possible, but if not, short base lines must be taken from S_1 and S_2, and observations made to C from both ends of these.

Referring now to Fig. 49, where C is the true station, S the satellite, it is clear that angle ACB = ADB − DAC = ASB + SBD − DAC = ASB + SBC − SAC. Hence to find the angle ACB from ASB we must *add* the value of SBC − SAC.

If circumstances permit it is desirable to observe the angles SBC and SAC directly from A and B.

But if this has not been done, their values can be calculated from the above data as follows :—

$$\frac{\sin \text{SBC}}{\sin \text{CSB}} = \frac{\text{SC}}{\text{BC}}$$

$$\therefore \sin \text{SBC} = \frac{\text{SC}}{\text{BC}} \times \sin \text{CSB}$$

$$\text{Similarly, } \sin \text{SAC} = \frac{\text{SC}}{\text{AC}} \times \sin \text{CSA}$$

All the factors on the right are known, as previously stated.

The observed angles at A always show how the other stations lie with respect to C, and a careful sketch should be made, as the formula for correction differs slightly in different cases.

Thus if SC lies between SB and SA, the formula becomes

$$\text{ACB} = \text{ASB} \pm (\text{SAC} + \text{SBC})$$

according to whether C is nearer to AB than S, or farther away. The student should have no difficulty in proving this case.

The angles ACB are then treated as observed angles for final correction as before, though somewhat less weight may be assigned to them, if desired, than to directly observed values.

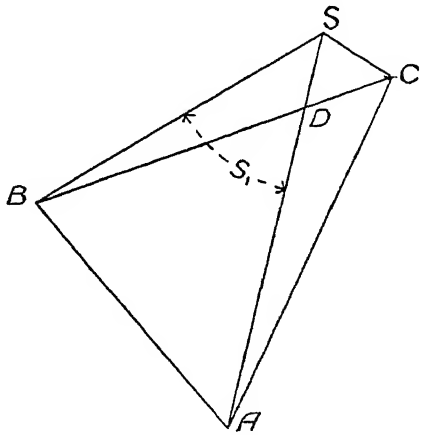


FIG. 49.

Intersected Points.—It is often convenient to determine intermediate points by observing *to them* from "trigonometrical points," without observing *at them*. Such points are called "intersected points." They should be determined from at least two triangles, and being inferior in precision to "trigonometrical points" they should not be used for any further work of triangulation. The position of "intersected points" can be determined with a degree of accuracy superior to chain measurement or traversing. They are especially convenient when the plan, or part thereof, has to be plotted to a scale so large that a sheet will not contain a sufficient

number of "trigonometrical points," or where the latter are not near enough together to prevent an accumulation of unavoidable error, appreciable in plotting. They also serve to fix the position of natural objects, such as spires, mill chimneys, etc. The computation of the "intersected points," is simple.

The *observed* values of the angles OAB, OAC (Fig. 50) may be obtained in two ways from the field book, by subtracting the mean bearing of the ray AB from that of AO, or AO from AC, and of BC from BO, or BO from BA. But though the sums of the two pairs of angles will equal the *observed* angles ABC and BAC, they will not necessarily be equal to the *corrected* angles adopted in triangulation. The difference between the sum of the two observed and the corrected angles must be taken, and half the difference

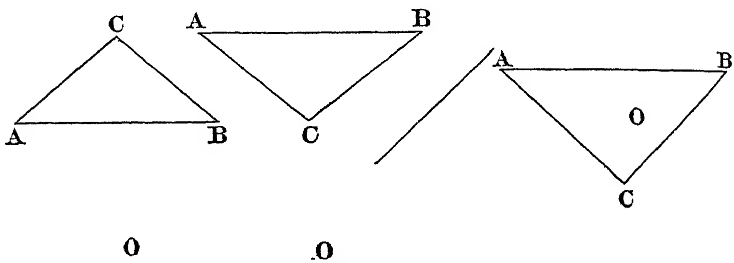


FIG. 50.

taken as the correction of each of the two angles, adding or subtracting so that the sum of the two corrected angles may be equal to the whole angle.

Then, deducing the value of the central angle at O (by deducting from 180°), determine OA, OB, and OC from the two best-conditioned triangles. Compute the "bearings" of these lines, and determine the "difference latitude" and "departure" of O from A, B, and C respectively. Then correct the "difference latitude" and "departure" found by small proportional corrections, so that the sum or difference of the "difference latitude" or "departure" of the "intersected point" from each "pair of points" agrees exactly with the "difference latitude" of each pair of points, and adopt the mean position so obtained.

The Three-point Problem.—It sometimes happens that one wishes to fix the position of a station by observations taken *from* it, instead of *to* it, as usual.

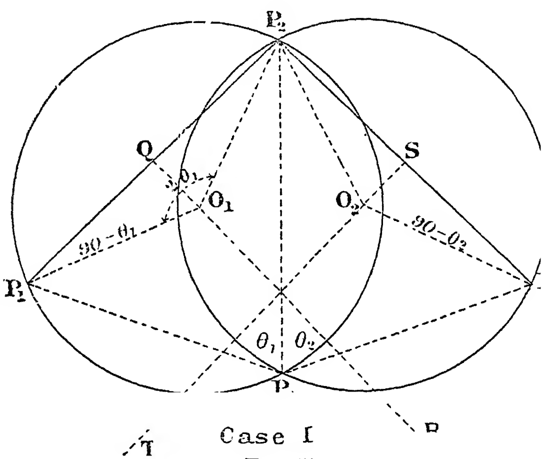
This arises, for instance, where a survey of certain detail, say by plane-table or tacheometer, is to be connected with a previous

triangulation. It is desired to fix a station near the detail withou the labour of taking the instrument all the way back to two c more triangulation stations.

The general method of fixing a station by observations from : is called *resection.*

For a direct solution, it is necessary that three known statior should be visible.

This problem may also often be of use in exploration work i hilly countries. For instance, having fixed the position of three c four conspicuous peaks, the surveyor may then fix the position (villages and other points as he proceeds on his journey. It is als



Case I

FIG. 51

useful in marine surveying to determine the position of points afloat.

Graphical Solution.—A graphical solution follows, and is drawn for three cases, showing different relative positions of the four points.

Let P_1, P_2, P_3 (Fig. 51) be the three points known by their ordinates, P_0 the unknown point at which the angles $P_1P_0P_2 = \theta_1$ and $P_2P_0P_3 = \theta_2$ are observed.

Bisect P_1P_2, P_2P_3 by perpendiculars QR and ST. At P_1 or P_2 lay off by protractor or natural tangent $\angle QP_1O_1 = 90 - \theta_1$.

Then O_1 is the centre of a circle passing through P_1P_2 and P_0.

For $\angle QP_1O_1 = 90° - \theta_1$ and $\angle P_1QO_1 = 90°$, $\therefore \angle QO_1P_1 = \theta_1$ and $\angle P_1O_1P_2 = 2\theta_1$.

But the angle at the centre of the circle is twice the angle at the circumference, $\therefore \angle P_1P_0P_2 = \theta_1$. Hence, any point on the circumference of a circle with centre O_1 and radius O_1P_1 or O_1P_2 subtends in the proper segment, the angle θ_1 at any point of the circumference.

Proceed similarly with P_2 and P_3, drawing through these points a segment of a circle containing the angle θ_2.

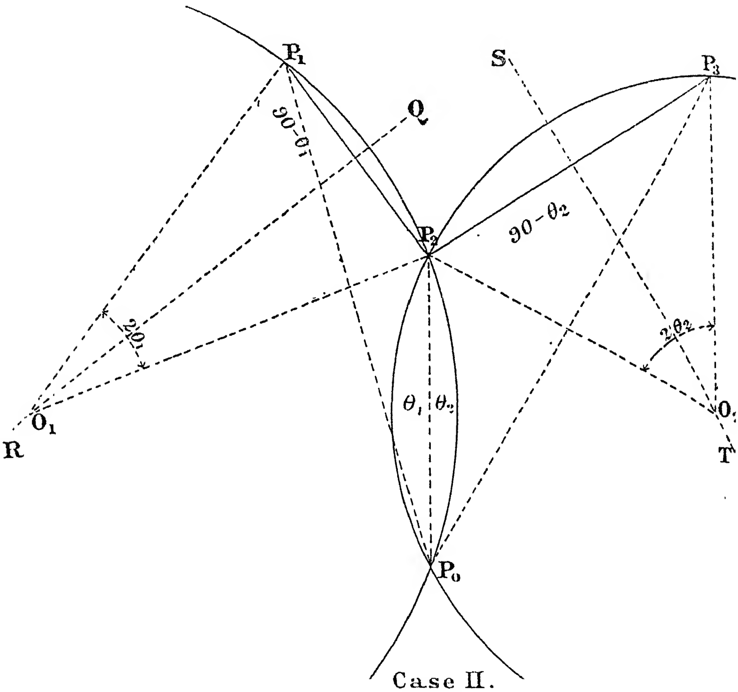


FIG. 52.

The intersection of these circles is the point sought, P_0.

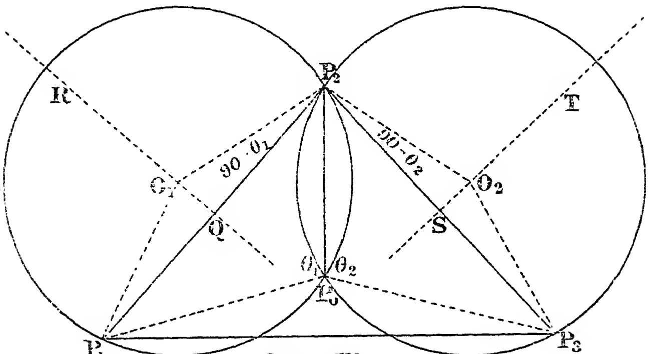
It will be seen that in all three cases a third circle may be drawn as a check through P_1 and P_3, using in cases I. and II. $\theta_1 + \theta_2$, and in case III. $360-(\theta_1 + \theta_2)$, as the angle to be laid off.

There are, therefore, three ways of solving this problem in each case, and the merit of this mode of construction is that it shows at a glance whether the data can give an accurate result, and

also which of the three solutions is that which will give the best determination.

Take case I. (Fig. 51). If P_1, P_2, P_3, and P_0 were so situated as to fall upon the circumference of a single circle, the problem is indeterminate; this can only happen when the points are so situated somewhat as in case I. It cannot occur when the points are situated as in cases II. and III. (Figs. 52 and 53).

It is therefore important that the three points observed should, if possible, be so situated with respect to P_0 as shown in cases II. and III., or, if not, at least the circle through P_1, P_2, and P_3 shall not pass near P_0. The more nearly it does so the less accurate the result.



Case III.
Fig. 53.

This is, of course, true whether the result be found graphically or by calculation.

The more nearly at right angles the two circles meet one another the better.

In case I. the intersection shown on the figure is fair, or at least as good as any other.

In the position of the points shown in case II., it so happens that the circle through P_1, P_0, and P_3 cuts either of the others more nearly at right angles than they cut each other. It would therefore be better to compute with the sides P_1P_2 and P_1P_3, or P_1P_3 and P_2P_3, than with those used in the construction.

Similarly in case III. as drawn, P_1P_2, P_1P_3 give the best intersection.

This consideration, however, really affects the selection of the final triangle to be used for the determination of the point P_0.

Thus in case I. (as drawn), both the triangles $P_1P_2P_0$ and $P_2P_3P_0$ are well conditioned.

In case II. (as drawn) the triangle $P_1P_3P_0$ is better than either of the others.

In case III. (as drawn), all these are pretty much the same, but on the whole it would be better to work from P_1P_2 and P_1P_3.

Solution by Calculation.—It is assumed that the co-ordinates of the three points P_1, P_2, and P_3 are known from the triangulation sheets, as well as the lengths and bearings of the lines joining them and the included angle between any two of them.

If only the co-ordinates are known, the rest can be calculated by the rules given in Chapter IV., Vol. I.

In Fig. 54, the angles θ_1, θ_2 are measured.

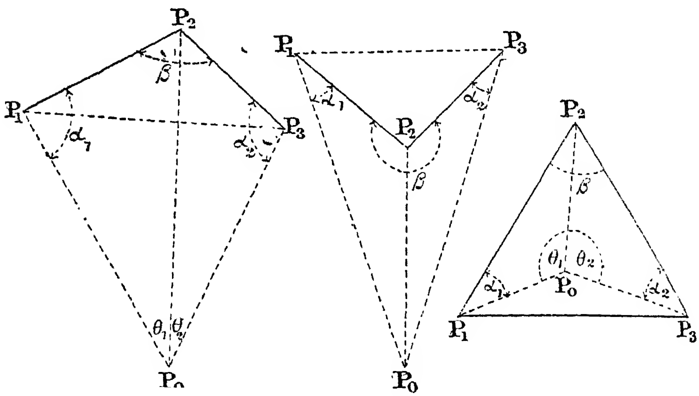


FIG. 54.

Let $P_3P_2P_1 = \beta$ (measured as shown in each case).

$\angle P_2P_1P_0 = a_1.$

$\angle P_2P_3P_0 = a_2.$

Then $\qquad \theta_1 + \theta_2 + a_1 + a_2 + \beta = 360°$

therefore $\qquad a_1 + a_2 = 360 - (\theta_1 + \theta_2 + \beta)$

$$= \text{a known quantity} \quad . \quad . \quad . \quad . \quad (1)$$

If $a_1 - a_2$ be determined, then the value of either a_1 or a_2 can be found.

But $\qquad P_1P_2 : \sin\theta_1 :: P_0P_2 : \sin a_1$

$$\therefore \frac{P_0P_2}{\sin a_1} = \frac{P_1P_2}{\sin \theta_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly,
$$\frac{P_0P_2}{\sin \alpha_2} = \frac{P_2P_3}{\sin \theta_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Divide (2) by (3). Then
$$\frac{\sin \alpha_2}{\sin \alpha_1} = \frac{P_1P_2 \sin \theta_2}{P_2P_3 \sin \theta_1} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the right-hand side involves only known quantities.

$$\text{Now put } \tan \phi = \frac{P_1P_2 \sin \theta_2}{P_2P_3 \sin \theta_1}, \text{ and find } \phi$$

$$\therefore \frac{\sin \alpha_2}{\sin \alpha_1} = \tan \phi$$

Adding and subtracting,
$$\frac{\sin \alpha_1 + \sin \alpha_2}{\sin \alpha_1 - \sin \alpha_2} = \frac{1 + \tan \phi}{1 - \tan \phi}$$

But by the rules of trigonometry
$$\frac{\sin \alpha_1 + \sin \alpha_2}{\sin \alpha_1 - \sin \alpha_2} = \frac{\tan \frac{1}{2}(\alpha_1 + \alpha_2)}{\tan \frac{1}{2}(\alpha_1 - \alpha_2)}$$

and
$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Remembering that $\tan 45^\circ = 1{\cdot}0000$, we may make $A = 45^\circ$ and $B = \phi$; then
$$\tan (45^\circ + \phi) = \frac{\tan 45^\circ + \tan \phi}{1 - \tan 45^\circ \tan \phi}$$

or
$$\tan (45^\circ + \phi) = \frac{1 + \tan \phi}{1 - \tan \phi}$$

Hence
$$\frac{\tan \frac{1}{2}(\alpha_1 + \alpha_2)}{\tan \frac{1}{2}(\alpha_1 - \alpha_2)} = \tan (45^\circ + \phi)$$
$$\cot \tfrac{1}{2}(\alpha_1 - \alpha_2) = \cot \tfrac{1}{2}(\alpha_1 + \alpha_2) \tan (45^\circ + \phi)$$

Hence $\alpha_1 - \alpha_2$ may be computed.
Comparing this with the value of $(\alpha_1 + \alpha_2)$ in equation (1), α_1 and α_2 are known.

Hence in each triangle we have two angles and one side, whence the remaining sides may be computed.

Again, by computing the remaining angles at P_1 and P_2 in the triangle $P_1P_2P_3$, the three angles of the triangle $P_1P_3P_0$ may be computed, as may be most expedient.

Example.—Let the ordinates of three points (Fig. 55) be as follows:

$$P_1 = 5534{\cdot}0\ \text{N.} \qquad 1845{\cdot}0\ \text{E.}$$
$$P_2 = 2486{\cdot}7\ ,, \qquad 1485{\cdot}0\ ,,$$
$$P_3 = 0000{\cdot}0\ ,, \qquad 0000{\cdot}0\ ,,$$

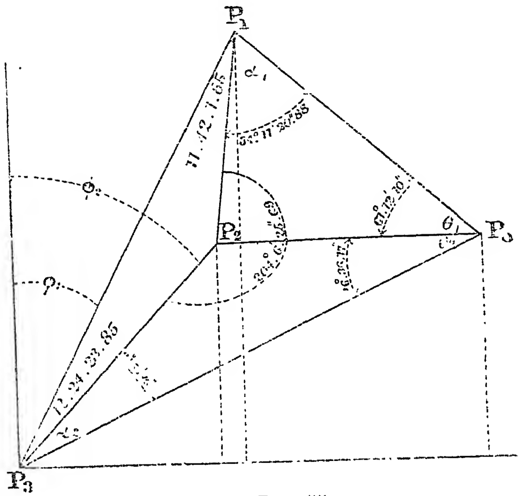


FIG. 55.

Let the following angles be observed at the point sought P_0.

$$\begin{aligned} P_1P_0P_2 = \theta_1 &= 61^\circ\ 12'\ 10'' \\ P_2P_0P_3 = \theta_2 &= 36^\circ\ 36'\ 17'' \\ \hline \theta_1 + \theta_2 &= 97^\circ\ 48'\ 27'' \end{aligned}$$

And suppose the following particulars are obtained from the triangulation sheets:—

Line.	Log of length.	Bearing.	Included angle.
		° ′ ″	° ′ ″
P_3P_1	3·7659258	18 26 17	12 24 23·85
P_3P_2	3·4618526	30 50 40·85	155 53 34·31
P_2P_1	3·4869248	6 44 15·15	11 42 1·84
P_3P_1	—	18 26 17	

The included angles are here given as the interior angles of the triangle, so that their sum should be 180°.

Hence $P_1P_2P_3$ (measured so as to face P_0) $= \beta = 360° - 155° 53' 34''\cdot31$
$= 204° 6' 25''\cdot69$

$$
\begin{array}{rrrr}
 & \circ & ' & '' \\
\beta = & 204 & 6 & 25\cdot69 \\
\theta_1 + \theta_2 = & 97 & 48 & 27\cdot00 \\
\hline
 & 301 & 54 & 52\cdot69 \\
 & 360 & 0 & 0\cdot00 \\
\hline
\alpha_1 + \alpha_2 = & 58 & 5 & 7\cdot31 \\
\frac{\alpha_1 + \alpha_2}{2} = & 29 & 2 & 33\cdot65
\end{array}
$$

$$
\begin{array}{lrrrl}
\text{Log} \sin \theta_1 = \log \sin & 61 & 12 & 10 & = 9\cdot9426677 \\
 & & & \log P_2P_3 & = 3\cdot4618526 \\
\cline{5-5}
 & & & & \ 13\cdot4045203 \\
 & & & \text{colog} & \ 6\cdot5954797 \\
\log \sin \theta_2 = \log \sin & 36 & 36 & 17 & = 9\cdot7754583 \\
 & & & \log P_1P_2 & = 3\cdot4869248 \\
\cline{5-5}
\log \tan \phi = \log \tan & 35 & 47 & 13\ 44 & = 9\cdot8578628 \\
 & 45 & 0 & 0 &
\end{array}
$$

$$
\begin{array}{lrrrl}
(\phi + 45) = & 80 & 47 & 13\ 44 & \\
\log \tan (\phi + 45) = \log \tan & 80 & 47 & 13\cdot44 & 10\cdot7899590 \\
\log \cot \frac{\alpha_1 + \alpha_2}{2} = \log \cot & 29 & 2 & 33\cdot65 & = 10\cdot2554855 \\
\cline{5-5}
\therefore \log \cot \frac{\alpha_1 - \alpha_2}{2} = \quad ,, & 5 & 8 & 47\cdot23 & = 11\cdot0454445 \\
\alpha_1 - \alpha_2 = & 10 & 17 & 34\cdot46 & \\
\alpha_1 + \alpha_2 = & 58 & 5 & 7\cdot31 & \\
\hline
2\alpha_1 = & 68 & 22 & 41\cdot77 & \\
\alpha_1 = & 34 & 11 & 20\cdot88 & \\
\alpha_2 = & 23 & 53 & 46\cdot42 &
\end{array}
$$

In the two triangles there are two angles and one side given, whence the other sides and the ordinates of P_0 may be computed as usual. In the present case it would, however, be better to calculate from the large and well-conditioned triangle $P_1P_0P_3$, of which the side P_1P_3 is known.

$$
\begin{array}{lrrrrrr}
 & \circ\ \ '\ \ '' & & \circ\ \ '\ \ '' & & \circ & '\ \ '' \\
\text{The } \angle P_1P_3P_0 = & (23\ 53\ 46\cdot42) & + & (12\ 24\ 23\cdot85) & = & 36 & 18\ 10\cdot27 \\
\text{The } \angle P_3P_1P_2 = & (34\ 11\ 20\cdot88) & + & (11\ 42\ \ 1\cdot84) & = & 45 & 53\ 22\cdot72 \\
\text{The } \angle P_1P_0P_3 = & & & & & 97 & 48\ 27\cdot00 \\
\hline
 & & & & \text{Sum} & 179 & 59\ 59\cdot99
\end{array}
$$

$$
\begin{array}{rrrr}
 & \circ & ' & '' \\
P_1 = & 45 & 53 & 22\cdot72 \\
P_0 = & 97 & 48 & 27\cdot00 \\
P_3 = & 36 & 18 & 10\cdot27 \\
\hline
 & 179 & 59 & 59\cdot99
\end{array}
$$

COMPUTATION OF TRIANGLE.

Triangle.	Angles.	Corrected angles.	Log sin.	Difference.	Sum.	Sides in feet.	
		° ′ ″					
$P_0P_1P_3$	P_1	45 53 22·72	9·8561247	$\bar{1}$·8601694	3·6260952	4227·61	P_3P_0
	P_0	97 48 27·00	9·9959553	—	3·7659258	5833·45	P_1P_3
	P_3	36 18 10·27	9 7724468	$\bar{1}$·7764915	3·5424173	3486·72	P_1P_0
		179 59 59·99					

$$\begin{aligned} &\qquad\quad\ \ ^\circ\quad\ \ '\quad\ \ '' \\ \text{Bearing of } P_3P_1 &= 18\quad 26\quad 17{\cdot}00 \\ \text{Add } P_1P_3P_0 &= 36\quad 18\quad 10{\cdot}27 \\ \text{Bearing of } P_3P_0 &= 54\quad 44\quad 27{\cdot}27 \end{aligned}$$

$$\begin{aligned} \log\sin 54^\circ\ 44'\ 27{\cdot}27'' &= 9{\cdot}9119828,\ \log\cos = 9{\cdot}7613826 \\ \log P_3P_0 &= 3{\cdot}6260952 \qquad\qquad 3{\cdot}6260952 \\ & \quad\ 3{\cdot}5380780 \qquad\qquad\ \ 3{\cdot}3874778 \end{aligned}$$

$\therefore$ Dep. = 3452·06, and diff. lat. = 2440·49
P_0 = 2440·49 N. 3452·06 E.

$$\begin{aligned} &\qquad\quad\ \ ^\circ\quad\ \ '\quad\ \ '' \\ \text{Bearing of } P_1P_3 &= 198\quad 26\quad 17{\cdot}00 \\ \text{Deduct } P_3P_1P_0 &= 45\quad 53\quad 22{\cdot}72 \\ \text{Bearing of } P_1P_0 &= 152\quad 32\quad 54{\cdot}28 \end{aligned}$$

$$\begin{aligned} \log\sin 152^\circ\ 32'\ 54''{\cdot}28 &= 9{\cdot}6637022,\ \log\cos = 9{\cdot}9481197 \\ \log P_1P_0 &= 3{\cdot}5424173 \qquad\qquad 3{\cdot}5424173 \\ & \quad\ 3{\cdot}206119\bar{5} \qquad\qquad\ \ 3{\cdot}4905370 \end{aligned}$$

Departure = 1607·38 E., diff. lat = 3094·12 S.
P_0 = 2439·88 N. 3452·38 E.

It is to be observed that when the sides and angles are given from the "triangulation sheets," the computation is not a formidable one. This method may therefore be useful in fixing intersected points from an existing triangulation.

The formula given is the neatest, and is suitable for every case; it is derived from "Adjustment of Observations," by T. W. Wright, B.A.

The following is, however, more easily remembered.

By equation (1), p. 130,

$$\begin{aligned} \alpha_1 + \alpha_2 &= 360^\circ - (\theta_1 + \theta_2 + \beta) \\ &= \text{A (say), where A is known} \\ \therefore\ \alpha_2 &= \text{A} - \alpha_1 \end{aligned}$$

and by equation (4)

$$\frac{\sin \alpha_2}{\sin \alpha_1} = \frac{P_1P_2 \sin \theta_2}{P_2P_3 \sin \theta_1}$$
$$= k, \text{ say, where } k \text{ is also made up of known quantities}$$

Both A and k can therefore be calculated at once.

Then
$$k = \frac{\sin \alpha_2}{\sin \alpha_1} = \frac{\sin (A - \alpha_1)}{\sin \alpha_1}$$
$$= \sin A \cot \alpha_1 - \cos A$$
$$\therefore \cot \alpha_1 = \frac{k + \cos A}{\sin A} = k \operatorname{cosec} A + \cot A$$

whence α_1 is easily found.

Mr. G. T. M'Caw, in a paper on "The Progress of Geodesy" (vol. xxxiii., *Proceedings of the Irish Institution of Civil Engineers*), gives a formula for finding the sides without computing the angles α_1 and α_2.

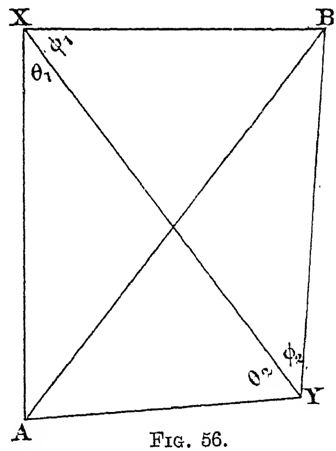


FIG. 56.

The Two-point Problem.—Although *one* point cannot be fixed by simple observation from it to *two* known points, yet if there are also two *unknown* points, *both* of them can be fixed by observations from each of them to one another, *and* to each of two known points.

Let A and B be the known points, X and Y the unknown points (Fig. 56).

Let the angles AXY $= \theta_1$ BXY $= \phi_1$ be observed, as also AYX $= \theta_2$, BYX $= \phi_2$. Assume some length for XY, say 1·000000 or 1000·0.

In the triangle XYA, the angles θ_1, θ_2 are given, and the base XY is assumed to be known.

Compute AX, AY in terms of XY, also in the triangle XYB compute BX and BY also in terms of XY.

Now in the triangle AXB you have two sides AX, BX known, in terms of the assumed values of XY, also the included angle $\theta_1 + \phi_1$. Hence calculate AB in terms of XY, also calculate the angles XAB and XBA.

Do the same in AYB.

To get the *real value* of the sides AX and AY, etc., you have merely to say, "As the calculated value of AB is to its true value, from triangulation, so is the assumed value of XY to its real value." So for the other sides. That is to say, you have to add to the logarithmic value of their computed lengths, the constant.

$$\text{Log}\left(\frac{\text{true value of AB}}{\text{calculated value of AB}}\right)$$

Use of two Trig. Points from any Existing Grand △n. in Lieu of a Base Line.—Hitherto it has been assumed that a base line

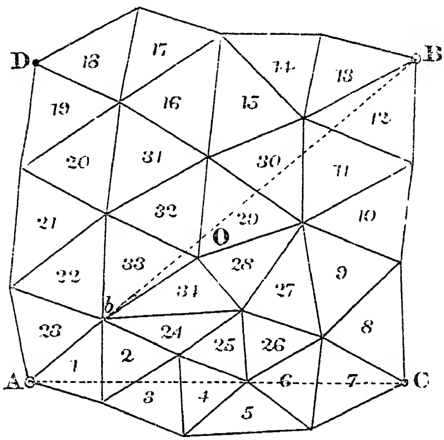


FIG. 57.

has been measured specially for the survey. It may, however, happen that a Grand Triangulation exists, and that the survey in question is undertaken for the purpose of completing the map. Any two points, A and B (Fig. 57), of the great trigonometrical network will serve as a base line, and it may be assumed that the distance between them is known far more accurately than any measurement can be made by the appliances obtainable by the surveyor.

Then by measuring a base A*b*, approximately, to serve as the known side for triangle No. 1 of the minor triangulation, and working round to the point B, we have only to get the length of AB from the minor triangulation in terms of the measured base A*b*. Then, as the value of AB, so found, is to its true value from the

main triangulation, so is the measured value of the base Ab to its true value, and all the calculated sides of the minor triangulation must be adjusted in the same proportion, the adjustment being made, as in the last case, by the addition or subtraction of a constant logarithm. Other points C and D of the great triangulation may be used as checks.

Other Forms of Problems.—The same problems may present themselves in modified forms. The following, for example, are included in the examinations in Geodesy in some of the colonies.

(1) Given three points A, B, C (Fig. 58), to fix the positions of two other

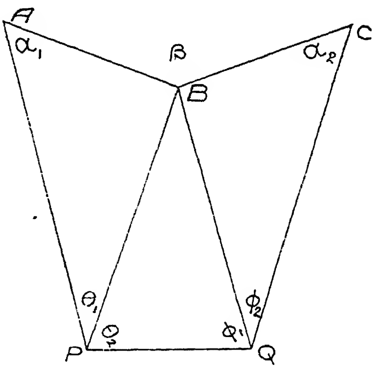


FIG. 58.

points, P and Q, by observing from P to A, B, and Q, and from Q to B, C, and P.

In the figure, AB, BC, and the angle β are given, and the angles $\theta_1, \theta_2, \phi_1, \phi_2$ are observed.

Put BAP $= \alpha_1$, QCB $= \alpha_2$.

Then
$$\frac{\sin \alpha_2}{\sin \phi_2} = \frac{BQ}{BC}$$

and
$$\frac{\sin \alpha_1}{\sin \theta_1} = \frac{BP}{AB}$$

$$\therefore \frac{\sin \alpha_1}{\sin \alpha_2} \times \frac{\sin \phi_2}{\sin \theta_1} = \frac{BP}{BQ} \times \frac{BC}{AB}$$
$$= \frac{\sin \phi_1}{\sin \theta_2} \times \frac{BC}{AB}.$$

$$\therefore \frac{\sin \alpha_1}{\sin \alpha_2} = \frac{\sin \theta_1 \sin \phi_1}{\sin \theta_2 \sin \phi_2} \times \frac{BC}{AB} = k, \text{ say}$$

Here all the terms on the right are known.

Again, the sum of the interior angles of the figure PQCBA is 540°

Hence
$$\begin{aligned}\alpha_1 + \alpha_2 &= 540° - (\theta_1 + \theta_2 + \phi + \phi_2 + 360° - \beta) \\ &= 180° + \beta - (\theta_1 + \theta_2 + \phi_1 + \phi_2) \\ &= A, \text{ say, where A is known}\end{aligned}$$

The problem therefore reduces to that of—given the sum of two angles and the ratio of their sines, to find the angles.

This has already been solved under the three-point problem (pp. 131 and 135).

When α_1 and α_2 are known, it is clear that all the triangles can be solved, and hence co-ordinates computed as before.

(2) Given four points A, B, C, and D, to fix two other points, P and Q, by observing from P to A, B, and Q, and from Q to P, C, and D.

In the figure, AB, BC, CD and the angles β_1 and β_2 are given, and θ_1, θ_2, ϕ_1 and ϕ_2 are observed.

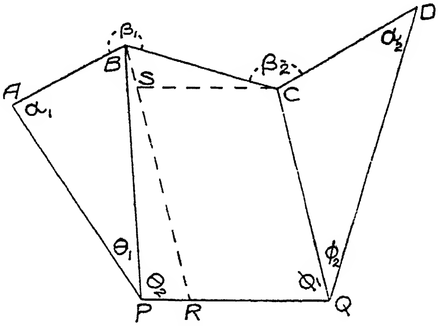


FIG. 59.

Put BAP $= \alpha_1$, QDC $= \alpha_2$.

Then
$$\frac{\sin \phi_2}{\sin \alpha_2} = \frac{CD}{CQ}$$
$$\therefore\ CQ = \frac{CD \sin \alpha_2}{\sin \phi_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now draw BR parallel to QC, and CS parallel to RQ. Therefore PRB $= \phi_1$ and BSC $= 180° - \phi_1$.

Then
$$\frac{\sin \alpha_1}{\sin \theta_1} = \frac{BP}{AB} = \frac{BR \sin \phi_1}{AB\ \sin \theta_2}$$
$$\left(\text{note } BP = BR \times \frac{\sin \phi_1}{\sin \theta_2}\right)$$
$$\therefore\ BR = AB \times \frac{\sin \alpha_1 \sin \theta_2}{\sin \theta_1 \sin \phi_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, in the triangle BSC, BS $=$ BR $-$ CQ

and the angle SCB $= 360° - \beta_2 - (180° - \alpha_2 - \phi_2) - (180° - \phi_1)$
$$= \alpha_2 + \phi_1 + \phi_2 - \beta_2$$
$$= (\alpha_2 + B), \text{ say}$$

where $B = \phi_1 + \phi_2 - \beta_2$, and is therefore known.

Hence
$$\frac{BR - QC}{BC} = \frac{\sin (\alpha_2 + B)}{\sin (180° - \phi_1)}$$
$$= \frac{\sin (\alpha_2 + B)}{\sin \phi_1}$$
$$\therefore\ BR - QC = \sin (\alpha_2 + B) \times \frac{BC}{\sin \phi_1} \quad . \quad . \quad . \quad . \quad (3)$$

Lastly, as before,

$$\alpha_1 + \alpha_2 = 720° - (360° - \beta_2) - (360° - \beta_1) - (\theta_1 + \theta_2 + \phi_1 + \phi_2)$$
$$= \beta_2 + \beta_1 - (\theta_1 + \theta_2 + \phi_1 + \phi_2)$$

$\therefore\ \alpha_1 + \alpha_2 = A$, say, where A is known . . . . (4)

In (1) put $\dfrac{CD}{\sin \phi_2} = m$

$\therefore\ CQ = m \sin \alpha_2$, where m is known.

In (2) put $\dfrac{AB \sin \theta_2}{\sin \theta_1 \sin \phi_1} = n$

$\therefore\ BR = n \sin \alpha_1$, where n is known.

Substitute these values in (3).

$$\therefore\ n \sin \alpha_1 - m \sin \alpha_2 = \frac{BC}{\sin \phi_1} \times \sin (\alpha_2 + B)$$

In this put $\alpha_1 = A - \alpha_2$, from (4), and also put $\dfrac{BC}{\sin \phi_1} = p$.

$$\therefore\ n \sin (A - \alpha_2) - m \sin \alpha_2 = p \sin (\alpha_2 + B)$$
$$\therefore\ n (\sin A \cos \alpha_2 - \sin \alpha_2 \cos A) - m \sin \alpha_2$$
$$= p (\sin \alpha_2 \cos B + \sin B \cos \alpha_2)$$

Divide by $\sin \alpha_2$.

$$\therefore\ n \sin A \cot \alpha_2 - n \cos A - m = p \cos B + p \sin B \cot \alpha_2$$
$$\therefore\ \cot \alpha_2 (n \sin A - p \sin B) = m + n \cos A + p \cos B$$
$$\text{or } \cot \alpha_2 = \frac{m + n \cos A + p \cos B}{n \sin A - p \sin B}$$

All the terms on the right are known, and hence α_2. All the other unknowns readily follow.

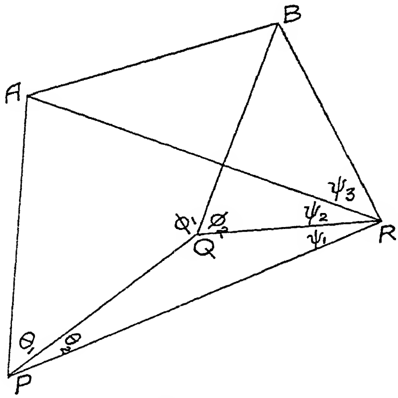


FIG. 60.

We divide by $\sin \alpha_2$ (rather than $\cos \alpha_2$), because from the conditions of the problem α_2 might be 90° (in which case $\cos \alpha_2$ would be zero), but α_2 could not be zero without this fact becoming known, hence $\sin \alpha_2$ cannot be zero.

(3) Given two points, A, B (Fig. 60), to fix three other points, P, Q, and

R, by observing from P to A, Q, and R; from Q to B, P, and R; and from R to A, B, P, and Q.

In the figure AB is given; θ_1, θ_2, ϕ_1, ϕ_2, ψ_1, ψ_2, ψ_3 are observed.

In this case choose some round number as an assumed length of PR, say PR $= p$.

$$\therefore \text{ in the triangle PQR, } QR = p\,\frac{\sin\theta_2}{\sin(\theta_2+\psi_1)}$$

$$\text{and in the triangle PRA, } RA = p\,\frac{\sin(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2+\psi_1+\psi_2)}$$

$$\text{and in the triangle BQR, } BR = QR\,\frac{\sin\phi_2}{\sin(\phi_2+\psi_2+\psi_3)}$$

$$= p\,\frac{\sin\theta_2\sin\phi_2}{\sin(\theta_2+\psi_1)\sin(\phi_2+\psi_2+\psi_3)}$$

Thus RA and BR are known in terms of p.

Now solve the triangle ABR from the two sides AR, BR and the included angle ψ_3, to find the angle ABR and the side AB.

The value of the angle ABR will be the *true* value, in spite of the incorrect length assumed for PR, as this simply alters the scale of the figure.

But the length of AB as calculated will differ from the known length, and all calculated sides must be increased or decreased in the ratio of

$$\frac{\text{true length of AB}}{\text{calculated length}}$$

The logarithm of this is worked out, and added as a constant correction to the calculated log of BR and the assumed log of PR.

The results give the true logs of those sides. AP is then easily found, and we have the data for calculating co-ordinates as usual.

If AQ be joined, we can find the length of that line in terms of p, and the angle AQB, by solving the triangle AQR, given QR, AR, and ψ_2.

Then we can find AB also from AQB, which gives a check.

Examples for Exercise.

(1) ABC is a triangle whose parts are known, as well as the co-ordinates of its corners. P is a point on the opposite side of AB from C. The angles BPA and PCB are observed. The co-ordinates of P are required. (Adapted from Survey Certificate Examination, Cape of Good Hope, 1904.)

Ans. From the known bearing of CB and the angle PCB, find the bearing of CP $= \theta$, say. Therefore, if y, x, be the North and East co-ordinates of P, c_1, c_2 those of C, then—

$$\frac{y-c_1}{x-c_2} = \cot\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Next P lies on a circle whose radius is $\frac{1}{2}$AB $\times \sec(\phi - 90°)$, where $\phi =$ BPA, and if O be the centre, ABO $= \phi - 90°$. Hence find the length and bearing of BO, and the co-ordinates of O, say, O_1, O_2. Then $(y - O_1)^2 + (x - O_2)^2 = r^2$, where r has the above value. From this and Equation No. 1, x and y can be found.

(2) AB is a base line, and C and D are inaccessible points close to the terminals of the base AB. It is required to determine the positions of C and D. Show by diagram how you would lay out the triangulation so as to find and check C and D, and what angles should be observed. (Same examination.)

(3) The length of a base AB is required. The lines AC_1 and CB, forming nearly a straight line, are measured. The angle C is very obtuse. Let k = no. of minutes in the supplement of angle C. $AB = a$; $BC = b$; $AB = c$. Prove that $c = a + b - 0{\cdot}000000042308 \times \dfrac{abk^2}{a+b}$ (see p. 152).
(Same examination, 1903.)

(4) In a triangle ABC the base $BC = a$ and the angle C are given. The angle B is measured, but is liable to an error the circular measure of which is θ. Show that the error ϵ corresponding to θ which will be produced in the side $AC = b$ is expressed very nearly by the formula

$$\epsilon = a\theta \frac{\sin C}{\sin^2 A}$$

(Same examination, 1902.)

(5) Referring to the triangulation net, Fig. 28, p. 84, suppose that NM and FG are measured bases, connected by the polygons with centres at N, R, and Q. The rest of the polygons are to be ignored. (*a*) Correct the three polygons simultaneously by the method of equal shifts. (*b*) If the logs of the measured bases are :—for MN 3·6619391 and for FG 3·4315492, compare the difference between these logarithms with the result of working through the chain of triangles, Nos. 7, 8, 9, 10, 11, 20, 21 (*vide* p. 79). Assuming that each base may be in error by the 100,000th part of itself, find the further corrections necessary to make the bases agree, and finally re-correct the rest of the triangles to make each polygon close. Triangles Nos. 1 and 2 are not to be touched.

Answer to Example 5.—(*a*) The equations are

$$94\cdot65x_N - 14\cdot05x_R = -1387$$
$$93\cdot65x_R - 14\cdot05x_N - 14\cdot9x_Q = +421$$
$$\text{and } 136\cdot8x_Q - 14\cdot9x_R = -154$$
$$\text{whence } x_N = -14\cdot3,\ x_R = +2\cdot2,\ x_Q = -0\cdot9$$

Angle . .	N_3	B_3	R_3	N_9	O_9	R_9	N_8	L_8	O_8	N_7	M_7	L_7
Correction(*a*)	+3·5	+15·8	−1·3	−9·7	−15·9	−2·4	−1·5	−13·6	+15·1	+7·8	−4·2	+24·4
Correction(*b*)	+1·8	+22·7	−6·5	−8·1	−15·9	−4·0	−1·5	−12·0	+13·5	+7·8	−2·6	+22·8

Angle . .	R_4	B_4	S_4	R_{12}	S_{12}	Q_{12}	R_{11}	Q_{11}	P_{11}	R_{10}	P_{10}	O_{10}
Correction(*a*)	+6·6	+ 4·6	+8·8	−0·7	+ 0·8	−3·1	−1·2	− 4·9	+ 4·1	−1·1	−3·0	+ 1·1
Correction(*b*)	+8·6	+ 5·1	+6·3	+2·6	− 2·4	−3·2	+0·4	− 6·5	+ 4·1	−1·1	−4·6	+ 2·7

Angle . .	Q_{13}	S_{13}	E_{13}	Q_{22}	E_{22}	F_{22}	Q_{21}	F_{21}	G_{21}	Q_{20}	P_{20}	G_{20}
Correction(*a*)	+5·1	+10·8	+9·1	+1·5	+ 7·1	+5·4	−1·5	+ 4·1	+ 2·4	+2·9	+6·6	+ 8·5
Correction(*b*)	+5·7	+11·1	+8·2	+1·0	+ 8·0	+5·0	+0·1	+ 2·5	+ 2·4	+2·9	+8·2	+ 6·9

(*b*) $\Sigma L - \Sigma R = 0\cdot2204350$, whereas log MN − log FG = $0\cdot2303899$. The discrepancy is 451, of which 86 can be left uncorrected for probable error of bases, and this necessitates a shift of 1·6″ from left to right. These being applied in the chain of triangles used, the corrections to No. 3 follow at once, as it is the only triangle left in first polygon. Polygons with R and Q as centres are then taken simultaneously. The trial corrections are found as before, except that the only triangles to be further adjusted are 4, 12, 13, and 22. R_{12} has a shift of $-\frac{1}{2}x_Q$, Q_{12} one of $+\frac{1}{2}x_R$. R_4 must get $+\frac{1}{2}x_Q$ to balance the angles at R. Hence S_4 must get $+x_R - \frac{1}{4}x_Q$ and B_4 must get $-x_R - \frac{1}{4}x_Q$, and so on. Finally, $x_R = +\cdot6$; $x_Q = -1\cdot5$, giving the values in the table.

CHAPTER IV

BASE LINE MEASUREMENTS, AND METHODS OF CHAINMENT WHERE GREAT ACCURACY IS REQUIRED

It is not possible, within the scope of this treatise, to give detailed descriptions of the methods hitherto adopted in base measurements, but it is considered desirable to describe, generally, how such measurements have been conducted, as well as the instruments used, and the degree of *absolute* (not *relative*) accuracy obtainable.

A base measurement being an absolutely necessary step in all triangulation (not in extension of previous surveys), a decision has to be made as to how it should be conducted in order to meet the following requirements. Firstly, the degree of accuracy aimed at, which is dependent on the object with which the survey is undertaken; secondly, the time available; and thirdly, the funds to hand.

The implements available, arranged in order of rapidity of manipulation, are as follows—

1. Metal chains, bands, or wires.
2. Wooden, glass, or metal rods.
3. Compensation bars, such as Colby's.

The first Geodetic Survey in England was started in 1783, when it was decided to connect Paris with Greenwich geodetically.

Hounslow Heath Base.—General Roy, appointed to carry out the work on the English side, commenced by measuring a base on Hounslow Heath in 1784.

The first measurement was made with a steel chain 100 feet long. It was considered merely experimental, and gave for the length of the line, after correction for temperature, 27,408·22 feet.

Deal rods had been generally used in other countries, accordingly three thoroughly seasoned trussed deal rods, each 20 feet 3 inches long, were next used. They were each terminated in bell-metal tips, by the contact of which measurement was made, and were compared with a standard scale. During the work it was

noticed that the rods were much affected by the changes of humidity in the atmosphere (there is no record that they were oiled or varnished), so the measurement was considered unsatisfactory.

The result of this measurement gave 27,406·26 feet for the length of the line, reduced to the level of the lower extremity at the temperature of 63° F., being that of the standard brass scale when the lengths of the deal rods were compared.

The base was next measured with glass tubes 20 feet in length, the expansions of which were found by experiment. The temperature of each tube was obtained during the measurement by two thermometers in contact with it.

This gave, when reduced to 62° F. and to the mean level of the sea, 27,404·0137 feet for the length of the line.

Reduction to M.S.L.—This reduction to M.S.L. (described on p. 80) was then used for the first time in the history of geodesy.

The greatest care was taken during each of the measurements to secure correct alignment, to adjust differences of level, to estimate variations due to temperature, and to allow for all other possible sources of error. Where each day's work left off a fine plumb-line was suspended to mark it off, the plummet vibrating in a brass cup, sunk in the ground and filled with water.

In 1791, when the work of the Ordnance Survey was resumed, it was decided to remeasure this base with a steel chain. Two chains of 100 feet in length were prepared by Ramsden. Each chain consisted of 40 links half an inch square in section, with brass handles flat on the under side, a transverse line on each handle marking the length of the chain. One chain was used for measuring, the other as a standard.

The chain, stretched by a weight of 28 lbs., was laid out in a succession of deal coffers, carried on trestles, the temperature being always taken during the measuring of each chain length.

Five thermometers were laid close by the chain in each position, and left till they showed nearly the same temperature, which took generally from 7 to 15 minutes.

The result of this measurement after reduction to 62° F. and M.S.L. was 27,404·24 feet, exceeding the length given by the glass tubes by 0·21 foot, and falling short of that by the deal rods by 2·02 feet.

These measurements are given in terms of General Roy's brass scale.

Before further describing base measurements, it is desirable to discuss the question as to what standard of length has been employed, not only in England but in other countries.

Standards of Length.—In measuring a base the length has to

be obtained in reference to some standard. The geodetic standards of length of different countries vary, both in form and in the material of which they are composed. They are divided into two classes, standards "*à traits*" and standards "*à bouts*," expressed by distances from centre to centre of points or scratches, or by end-to-end measurement. In the first, the lines or dots defining the measure are engraved on small discs of silver, platinum, or gold let into the bar. In the second, the bar generally has its extremities in the form of a small cylinder, presenting a circular disc, either plane or convex, of hard polished metal or sometimes of agate, for the contact measurements.

Different Foreign Standards.—The unit of the length in which by far the greater part of the geodetical measurements of Europe are expressed is the *Toise of Peru*, a measure "*à bouts*," of which fortunately there exist two copies (compared with the original and certified by Arago), one made for Struve in 1821, the second for Bessel in 1823. The standards of Belgium and Prussia are copies of the toise of Bessel. The Russian standard, which is two toises in length, is measured from the toise of Struve.

O.S. Standard.—The standard of the Ordnance Survey is 10 feet in length (deduced from the standard yard), and is in section a rectangle of $1\frac{1}{2}$ inches in breadth by $2\frac{1}{2}$ in depth, supported on rollers at a quarter and three-quarters of its length. The ends of the bar are cut away to half its depth, so that the dots marking the measure of 10 feet are in the neutral axis.

Standard Yard.—The standard yard of this country and its copies are bars 1 inch square in section, of iron, steel, brass, or copper. The lines defining the yard are in the axis of the bar. The length of the bar is that at 62° F., and is fixed by Act of Parliament which declares that "the pendulum vibrating seconds of time in a vacuum in the latitude of London, at the level of the sea, is 39·1393 inches of the standard, and that the yard shall be in the proportion of 36 to 39·1393 inches."

Base Measuring Instruments.—Mention has been made of the use of chains, glass, and deal rods in the measurement of the Hounslow Heath base. The steel chain has been found to give very good results, and has the great advantages of simplicity, portability, and cheapness, which will render it advisable to use it in countries where transport is difficult. But when extreme accuracy is required, in order to evade the temperature difficulty different forms of measuring rods have been devised.

Borda's Rods.—The apparatus used by the French, and constructed by Borda, consisted of two strips of metal in contact, forming a metallic thermometer carried on a stout beam of wood.

The lower strip was of platinum two toises in length, and lying immediately on it was a strip of copper about 6 inches shorter, fastened to it at one end only, so that it was free to move along the platinum as its relative expansion required. A graduated scale at the free end of the copper, and a corresponding vernier on the platinum, indicated the varying relative lengths of the copper, whence were inferred the temperature and length of the platinum strip. At the free end of the latter, uncovered by copper, was a graduated slider moving in a groove, which was used to measure the interval between two successive platinum strips.

Struve's Bars.—The Russian astronomer, F. W. Struve, used a wrought-iron bar two toises in length, one end terminating in a small steel cylinder, its end being slightly convex and highly polished. The other end carried a contact lever of steel, the lower arm of which terminated in a polished hemisphere, and the upper arm traversed a graduated arc also rigidly connected with the bar. The length of the bar was known to whatever division of the arc the index line at the end of the lever pointed. In measuring, the bars were brought into contact, which was maintained by a spring action on the lever. Two thermometers, whose bulbs were let into the body of the bar, showed its temperature.

The probable errors of the seven bases measured with these bars range from $\pm 0{\cdot}73\mu$ to $\pm 0{\cdot}914\mu$, where

$$\mu = \text{a millionth part of the length measured.}$$

Bessel's System.—The Germans used Bessel's system, in which the platinum and copper of Borda were replaced by iron and zinc, and the intervals were measured with a glass wedge. The upper or zinc rod terminated at either end in a horizontal knife-edge. The rods were supported on seven pairs of rollers carried by a bar of iron.

The probable error of Bessel's base was found to be $\pm 2{\cdot}2\mu$.

This apparatus was used in the Belgian bases measured (1852-53) by General Nerenburger with every precaution, particularly as to fixing the end of each day's work most minutely.

The mean errors were then computed to be, for the Beverloo base 2300 metres in length, $\pm 0{\cdot}59\mu$, and for the Ostend 2488 metres in length, $\pm 0{\cdot}45\mu$.

U.S.C. Survey Apparatus.—The United States Coast Survey Base Apparatus, devised by Professor Bache in 1845, combined the principle of Borda's measuring rods, the compensation tongue of Colby's, and the contact lever of Struve's. The cross-sections of the bars were so arranged that, while they had equal absorbing surfaces, their masses were inversely as their specific heats,

allowance being made for their difference of conducting power. The components were placed edgeways, the iron above and the brass below firmly united together at one end. The brass bar, which had the largest cross-section, was carried on rollers mounted in suspending stirrups, and the iron bar rested on small rollers which were fastened to it and ran on the brass bar. Supporting-screws through the sides of the stirrups retained the bars in place. The connection between the free ends of the component bars was the lever of compensation, which was pivoted to the lower bar. A knife-edge on the inner side of this lever abutted against a steel plane on the end of the upper or iron bar. At its upper end this lever terminated in a knife-edge facing outwards, in a position corresponding to the compensation points in Colby's bars. The knife-edge pressed against a collar on a sliding rod moving in a frame affixed to the iron bar above. The sliding rod was drawn backwards by a spiral spring, through which it passed and kept the lower knife-edge of the lever pressed with constant pressure against the iron bar. The sliding rod terminated in an agate plane for contact. A vernier attached to this end of the bar gave their difference of length as a check on the work.

At the other end, where the bars were united, there was a corresponding sliding rod terminating outwardly in a blunt horizontal knife-edge. The inner end abutted against a contact lever pivoted below. This lever, when pressed by the sliding rod, came in contact with the short tail of a level mounted on trunnions and not balanced. For a certain position of the sliding rod the bubble came to the centre, and this position gave the true length of the measuring bar. This was an exceedingly delicate mode of measuring.

At this end of the apparatus there was also a sector for indicating the inclination of the bar in measuring, and it is to the arm of this sector that the contact-lever and level were attached.

As in Colby's apparatus, however, the compensation cannot always be absolutely relied on. The length of the bar depended on whether the temperature was rising or falling, and a length had to be assigned from actual comparisons in each of these conditions.

Eight or more bases have been measured with these bars, which offered considerable facility for rapid work, as much as a mile in one day having been completed with them.

One of the last bases, that of Atalanta in Georgia, was measured twice in winter and once in summer, the temperatures extending from 18° F. to 107° F., by which means an extreme test of the performance of the bars was afforded.

The probable error of this last base was $\pm 1 \cdot 76\mu$, whilst those of the seven previously measured varied from $\pm 1 \cdot 8\mu$ to $2 \cdot 4\mu$.

Porro's System.—A wholly different system was that of Porro. Only one measuring bar was used, composed of two cylindrical rods of steel and copper laid side by side, firmly united at their common centre and free to expand outwards. With this bar the successive equal intervals between microscopes arranged in the line of the base were measured.

A modification of Porro's apparatus, as made by Colonel Hassard, was used for three bases in Algiers, giving probable errors of $\pm 1 \cdot 0\mu$ in each case.

Colby's Compensation Bars.—On the Ordnance Survey, the Compensation Apparatus invented by Major-General Colby, already alluded to, was used.

The upper bar was of iron, the lower of brass, 10 feet in length, firmly connected at their centres. At either extremity was a metal tongue about 6 inches long, pivoted to both bars, so as to be perfectly firm and immovable while yet not impeding the expansion of the bars. A silver pin let into the end of each tongue carried a microscopic dot marked c, c' (*vide* Fig. 61). The letters ab, $a'b'$ refer to the axes of the pivots shown by dots.

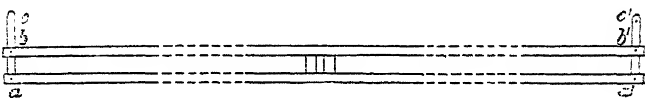


FIG. 61.

Now if α, β be the rates of expansion of the brass and iron bars aa', bb' respectively, by construction

$$ac : bc = \alpha : \beta = \quad : b'c'$$

Now the centres of the bars being fixed, let an increase of temperature imparted to the bars cause a to move off the small distance αi, while b is carried in the same direction the amount βi. It is clear that the movement of c is zero, that is, the distance of the dots c, c' remains 10 feet.

In order to ensure the proper action of this mechanism, the radiation and absorption of heat by the bars must be equal. This was effected by clouding and varnishing the surfaces until by experiment the rates of heating and cooling were found to be the same.

This compound bar was placed in a deal box (resting immediately upon two brass rollers in the bottom of it), and kept from moving by means of a pin fixed in the bottom of the box.

The complete set contained six bars. Each box, when in use, was supported at a quarter and three-quarters of its length by strong brass tripods having rollers on their upper surfaces.

The interval between two adjacent bars was exactly 6 inches, and was measured by a "compensation microscope."

The end of each series of six bars in the measurement was transferred to the ground by means of a "point carrier," which was a massive triangular plate of cast iron, having attached to its surface, or at a height above it that may be varied as required, an adjustable horizontal disc with a fine point engraved on it. This point was adjusted to bisection in the focus of the advanced telescopic microscope.

The Lough Foyle and Salisbury Plain bases, and ten bases in India, have been measured on this system.

Unfortunately, these bars have not given unqualified satisfaction, especially in India, where they were tested on a portion of the Cape Comorin base.

The probable error in the measurement of a base line with this apparatus is about $\pm 1{\cdot}5\mu$.

Lough Foyle Base.—On the Ordnance Survey, Colby's bars were first used in the measurement of the Lough Foyle base in 1827–28.

Salisbury Plain Base re-measured.—In 1849 it was determined to re-measure the Salisbury Plain base with the same apparatus, it having been measured with a chain in 1794.

Salisbury Plain is very well adapted to the measurement of a base, and a longer line might have been selected on it, but it is certain that little, if any, advantage is gained by the measurement of a base of more than 6 or 7 miles, provided it be surrounded with very careful triangulation.

Comparison of Bars with O.S. Standard.—The bars were most carefully compared with the Ordnance Survey standard. The standard was in every case first brought under the microscope, then the six compensation bars in succession, and then the standard again. The temperatures of the bars, as well as the readings of the micrometer, were registered. The standard bar has the bulbs of two mercurial thermometers let into it, and the interstice being filled with oil, the effect of the change in temperature of the air is avoided. Some 65 sets of comparisons were taken during the measurement of the base.

Result.—After all the necessary corrections, etc., were made, the length of the line stood thus—

Measured with compensation bars .	34,840·8579
,, ,, intermediate compensation microscopes	1,741·9234
,, back with beam compass (from nearest multiple of bars).	4·2938
Reduction to level of sea	0·6294
Length of base line in feet of Ordnance Survey standard	36,577·8581

Broken Base.—It is always best to measure a base in a straight line between the extremities, but on account of the difficulty in finding a suitable place of sufficient length, it is sometimes necessary to measure a "broken base," as was done in the bases of Melun and Perpignan.

In Fig. 62, suppose ACB to be the broken base, of which the lengths of AC and CB are known, also the angle ACB, which may be supposed to be nearly 180°.

Let BCI $= \theta$, a very small angle.
ACB $= 180° - \theta$.
CB $= a$.
AC $= b$.
AB $= c$, the length of base required.

Then $$c^2 = a^2 + b^2 + 2ab\cos\theta$$

and as θ is very small,

$$\cos\theta = 1 - \frac{\theta^2}{2}$$

$$\therefore\ c^2 = a^2 + b^2 + 2ab\left(1 - \frac{\theta^2}{2}\right)$$
$$= a^2 + b^2 + 2ab - ab\theta^2$$
$$= (a+b)^2 - ab\theta^2$$

whence $$c = (a+b)\left\{1 - \frac{ab\theta^2}{(a+b)^2}\right\}^{\frac{1}{2}}$$

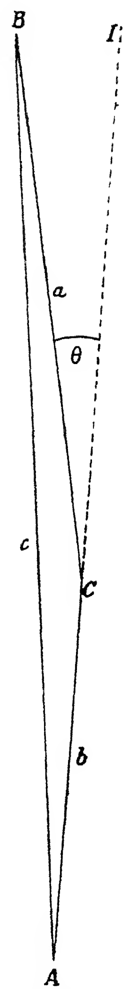


Fig. 62.

Expanding this series and letting θ denote the number of minutes in the angle,

$$c = a + b - \frac{ab\theta^2 \sin^2 1'}{2(a+b)} \text{ (very nearly)}$$

that is
$$c = a + b - \frac{ab\theta^2}{a+b} \times \frac{4{\cdot}2308}{10^8}$$

Again, to find the angle BAC or A,

$$\frac{\sin A}{\sin \theta} = \frac{a}{c} \text{ and } \sin \theta = \theta - \frac{\theta^3}{6}$$

$$\therefore \sin A = \frac{a\theta}{c}\left(1 - \frac{\theta^2}{6}\right)$$

Substituting above value for c,

$$\sin A = \frac{a\theta}{(a+b)\left\{1 - \frac{ab\theta^2}{(a+b)^2}\right\}^{\frac{1}{2}}} \times \left(1 - \frac{\theta^2}{6}\right)$$

$$= \frac{a\theta}{a+b}\left\{1 + \frac{ab\theta^2}{2(a+b)^2} - \frac{\theta^2}{6} - \frac{ab\theta^4}{12(a+b)^2}\right\}$$

Neglecting θ^4, etc.

$$= \frac{a\theta}{a+b}\left\{1 + \frac{3ab - a^2 - 2ab - b^2}{6(a+b)^2}\theta^2\right\}$$

$$\therefore \sin A = \frac{a\theta}{a+b}\left\{1 + \frac{ab - a^2 - b^2}{6(a+b)^2}\theta^2\right\}$$

Now
$$A = \sin A + \frac{\sin^3 A}{6}$$

Substituting this value for sin A, and putting the small angles A and θ equal to A sin 1′ and θ sin 1′ to express them in minutes, we have

$$A = \frac{a\theta \sin 1'}{a+b} + \frac{ab(a-b)\theta^3 \sin^3 1'}{6(a+b)^3}$$

Prolonging a Base.—It is sometimes desirable to increase the length of a measured base, over ground which does not lend itself conveniently to direct measurement, owing to some kind of obstacle, such as a river, ravine, or broken ground, intervening. In such a case the completed measurement may be prolonged to the desired position of the base terminal on the following principles :—

In Fig. 63, suppose AC to be a base line of which AB has been measured, E and D having been fixed during the measure-

ment. Select stations F and G so that the angles at E may be nearly right angles and the points themselves nearly equidistant from the line and about equal to AE. Similar conditions determine the position of H, I, K, and L. At A and all the other points along the line, and on either side, all the points visible are

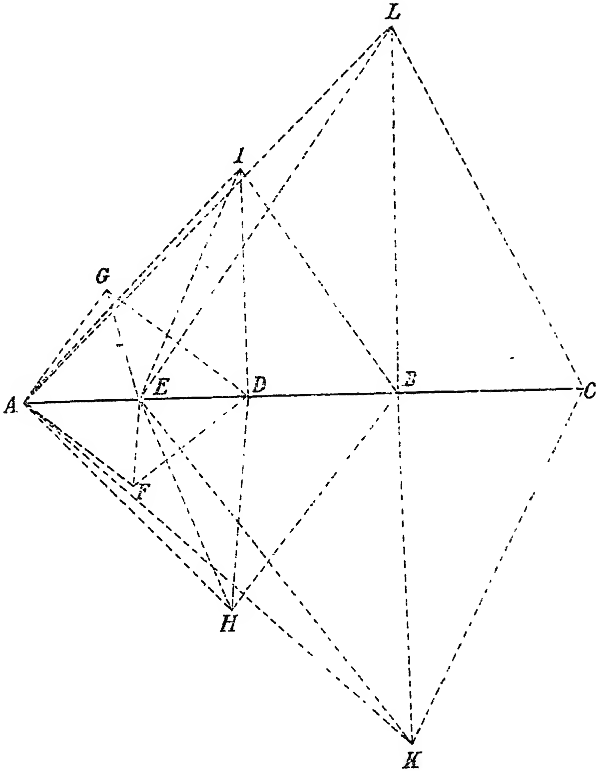


Fig. 63.

observed with a theodolite. From AE and the observed angles, EF and EG are determined, from each of which in the triangles DEF, DEG, the side ED is obtained, the distances thus found forming two checks on the measured length. DH and DI are in like manner calculated from AD and ED as bases, and each of these again furnish data for the determination of DB. Lastly, BL and BK are found from AB and also from EB.

From the mean results of these BC is obtained, thus giving an addition to the measured portion of the base. This latter operation is called the "prolongation of a base," and is often convenient or necessary in order to complete or extend a base.

The measured portion of the Lough Foyle base was about 8 miles, and 2 miles were afterwards added by this method.

By Subsidiary Bases.—The prolongation of a base may also be performed by the help of "subsidiary bases," measured for that purpose. Suppose, for instance, in the last figure that CE had been measured, then EF and EG would be measured as subsidiary bases, and the length EA calculated by using them and the angles observed from E, F, G, and A.

For another method of extending a base line into a longer line, see p. 88.

Estimates of Accuracy.—All estimates of the accuracy of base measurements must be accepted with reserve, for they must necessarily be relative, and comparison is frequently made between two or more measurements made with the same instruments and the same staff over the same ground, the result being, not the approximation to absolute accuracy, but the *difference* between certain measurements which may or may not be accurate, either as regards their units or the aggregation of these units.

In Gore's "Elements of Geodesy" (New York, 1886), the accuracy obtainable in base line measurements is stated to be as high as 1 in 6,000,000.

Jäderin Wires.—In or about the year 1883, Professor Jäderin of Stockholm worked out a method of measuring a base by means of two wires of different metals (steel and brass) stretched between tripods.

Each wire was provided at its forward end with a graduated scale. The tripods were carefully aligned, and placed as nearly as possible at the correct distance apart before stretching the wires. Then, the back end of each wire being brought to coincide with a mark on the head of the back tripod, the reading of the corresponding mark on the forward tripod is taken on the scale attached to each wire, and so on.

A constant tension was, of course, applied to the wires.

The idea of the two wires was similar to that of Borda's rods, already described.

That is, assuming that the wires were of exactly the same length at some standard temperature, then the farther removed from this the temperature in use became, the greater the difference in length, as the coefficients of expansion are unequal.

Hence, also, the greater would be the difference between the

lengths of the base as given by the two wires, uncorrected for temperature.

Thus this difference was in itself a measure of the actual temperature of the wires. It has already been explained that this temperature does not, in general, agree with that of the air, and is difficult to find.

Jäderin wires enabled a base to be measured much more quickly than with the older measuring rods or bars, and with an accuracy probably not much inferior, or quite as good, as most of these.

Invar.—In or about the years 1897–98, the application of the newly discovered alloy called ***invar*** to base line measurements was suggested.

This application, together with a full account of the properties of invar, the construction of the wires, and the examination to which they should be submitted before use, and a general account of the necessary corrections to measurements made with wires, is contained in a book called "La mesure rapide des bases géodésiques," par J. René Benoit, and Ch. Ed. Guillaume, of the Bureau international des poids at mesures.

M. Guillaume was the inventor of invar, which is an alloy of 64 per cent. of steel and 36 per cent. of nickel, having a very small coefficient of expansion indeed.

It is well known that iron exists in different states known as *beta* and *gamma* iron; the change from *beta* to *gamma* takes place at a bright red heat, and is accompanied by a sharp contraction.

It appears (*vide* remarks by Mr. Jackson in a discussion on "The Progress of Geodesy," *Proceedings of the Irish Institution of Civil Engineers*, vol. xxxiii., 1907) that, when alloyed with nickel in the above proportions, the change of state takes place at ordinary atmospheric temperatures, and the consequent contraction almost completely neutralizes the thermal expansion.

The alloy itself has, however, apparently some molecular instability. The effects of this, and means for overcoming it, are fully set forth in the above book.

When properly used, it seems that the coefficient of expansion can be relied upon not to exceed 0·5 part per million per degree Fahrenheit, so that an error of at least 2° F. is necessary to affect the result to the extent of 1 part in 1,000,000.

With a steel wire the true temperature must be known accurately to nearly 0·1° F. to avoid a similar error.

A diameter of 1·65 millimetre is recommended for the invar wires, and they are suspended from tripods, and fitted with scales, like the Jäderin wires.

Very elaborate appliances are used at the Bureau at Sèvres to standardize the wires and find their exact lengths at some known temperature and tension, which is one of the most difficult and important parts of the work.

Base at Lossiemouth.—In 1909 a geodetic base line was measured near the town of Lossiemouth in Elgin, Scotland, by the British Ordnance Survey department.

A full account of this measurement is contained in the official account, No. 1 of the new series professional papers issued by the department.

For this measurement invar tapes were used instead of wires. The tape has the disadvantage of offering a larger surface to any wind, which not only blows the tape into a horizontal curve, thus altering the apparent length, but causes vibration which makes it difficult to read the scale at the end.

On the other hand, it is quite easy to detect any twist or torsion of the tape by simply walking along it, whereas with a wire one complete twist would appreciably alter the length, and this requires special precautions to detect it. Moreover, it is easier to fix the scales at the ends firmly to a tape than to a wire.

In the measurement at Lossiemouth, the tapes were suspended between tripods, whose heads were adjustable in any horizontal direction and carried the terminal mark. These tripods were accurately lined in by means of a theodolite, and placed as nearly as possible at the correct distance apart, so that it would always be possible to read the exact positions of the terminal marks on the scales at each end of the tape. These scales were only one inch long on each side of the true end of the tape, and divided into 25ths of an inch. About eight readings were usually taken for each span at each end, the tape being slightly moved one way or the other between each pair, so as to alter the readings.

The tension was applied by weights passing over frictionless pulleys carried by "straining trestles" or tripods, placed accurately in the line of the tape produced, and so adjusted in level as to bring the graduated part of the tape just into contact with the terminal mark on the tripod.

After each six spans the observers changed ends, to eliminate the effect of personal bias in taking the readings.

The temperature was taken for each span. A small error in this is negligible for invar, as already stated.

At the end of each day's work, the tripods with the marks were left in position for next day's work. But to avoid mishap, the last terminal mark was transferred to the ground by two theodolites at right angles, and marked on a copper plate fixed on

the ground. It was found that the maximum error likely to occur in transferring a point in this way was $\frac{1}{120}$th of an inch.

For a full account, with illustrations, the reader is referred to the above pamphlet.

Corrections.—The corrections to be applied include those for temperature, slope and sag, which have already been referred to in Part I. of this work, Chap. IV., p. 235, and for reduction to sea-level, for which see p. 80 of this volume.

One of the most difficult parts of the work is the standardization of the tape to make sure of its exact length, with an error less than 1 part in 1 million.

Considerable space is devoted to this in the official account, and six tapes were used altogether, viz. two standard 100-foot tapes which were standardized at Southampton, and three 100-foot field tapes which were compared with the standards in the field, as well as one 300-foot field tape. As the straining was done by weights, it was necessary to use a slightly different weight in the field, so as to allow for the change of the force of gravity between Southampton and Lossiemouth, and thus ensure the same tension. The formula for calculating this is contained in the account, as well as a specially contributed article dealing with the correction for sag, slope, etc., by Professor O. Henrici and Captain E. O. Henrici.

The tape should, for the best work, of course be standardized under the same conditions as in use, and *not* lying flat, and the formulæ for the corrections must be more accurate than the approximations given in Part I. of this work.

The following is the formula arrived at :—

Let S = *nominal* length of tape, say 100 feet.

a_0 = error of the tape, determined when standardized, *plus* if the tape be too long.

a_1 = the sum of the readings at the ends of the tapes, being the total excess in the length of tape used over and above the length between the terminal marks on the tape ; thus the readings are to be reckoned as *plus* if *outside* of the terminal marks, and *minus* if inside.

α = coefficient of thermal expansion.

t° = excess of temperature above standard.

Then put $A = a_0 + a_1 + S\alpha t$

$$\text{Also put } P = S\left(\frac{1}{2}\cdot\frac{h^2}{S^2} - \frac{1}{8}\cdot\frac{h^4}{S^4} + \frac{1}{16}\cdot\frac{h^6}{S^6} + \frac{5}{64}\cdot\frac{h^8}{S^8}\right)$$

$$Q = A\left(\frac{1}{2}\cdot\frac{h^2}{S^2} + \frac{3}{8}\cdot\frac{h^4}{S^4} + \frac{5}{16}\cdot\frac{h^6}{S^6}\right)$$

and

$$R = \lambda\left(\frac{h^2}{S^2} + \frac{1}{2}\cdot\frac{h^4}{S^4} + \frac{3}{8}\cdot\frac{h^6}{S^6}\right)$$

where h = difference in level between the tripod marks, and λ = apparent shortening of tape due to sag. Then if X = horizontal distance between the tripod marks,

$$X = S - P + A + Q + R$$

This formula is stated by the authors to be correct to one part in ten million if $\frac{h}{S}$ be less than $\frac{1}{10}$ and A be less than $\frac{S}{1000}$, and the tension applied be greater than 20 times the weight of the tape, provided that the mean tension is kept constant.

The formula given for λ is

$$\lambda = \frac{X_0{}^3 w^2}{24T^2}$$

where $X_0 = S + A$,
w = weight of tape (or wire) per unit length,
T = tension applied.

This formula is stated to be accurate within one part in 10,000,000 if T be greater than 20 times the weight of the tape or wire.

For proofs and further particulars the reader is referred to the official account.

Results.—The result of this base was 23,525·279 feet, with a probable error of $\frac{1}{900000}$.

The rate of measurement was about 1200 feet per day at first, increasing to twice that amount as the work proceeded.

The average distance per day in measuring the Lough Foyle base with Colby's compensation bars was about 250 feet, and the probable error was slightly greater than the above.

South African Survey.—An interesting paper on "The Progress of Geodesy," by G. I. M'Caw, Esq., is contained in the *Transactions of the Irish Institution of Civil Engineers* for 1907, vol. xxxiii. In addition to a general review of the subject, it gives much valuable information about the measurement of base lines, an original solution of the three-point problem, and a diagram of the triangulation of South Africa.

CHAPTER V

PRACTICAL ASTRONOMY

Need of Knowledge.—A knowledge of practical astronomy is necessary for the following purposes in surveying :—

(*a*) To find the latitude of a station. This is required not only for determining the true shape and dimensions of the earth (p. 222), but also for showing the positions of places on its surface, for checking extensive surveys, and so on.

(*b*) To find the direction of the meridian, or true north and south line, at any station.

(*c*) To determine time and longitude.

The Earth's Movement.—To properly understand the methods used in these determinations, some preliminary study of the earth's movement is essential.

It is well known that the earth rotates round its polar axis, producing the effect of night and day, while at the same time it revolves in the same direction round the sun, completing one revolution in 1 *year*.

Meridian.—The plane which contains the earth's polar axis, and any other point on the surface, is called the *meridian*, or *meridian plane*, of that point.

As the earth rotates on its axis, each meridian plane is carried round from west to east, and passes through each heavenly body in turn, completing one rotation in about one *day*.

The meridian of any point meets the earth's surface in a great circle passing through the point and the poles. Hence it defines the true *north-and-south* line through the point. Thus if PCQ (Fig. 64) be the meridian of C, CP is due north, and CQ due south.

At the moment when the meridian plane of a point passes through any heavenly body, therefore, that body is due north or south of the point.

As the meridian of a point C (Fig. 64) makes a complete circle, PCQD, round the earth, it clearly passes through each heavenly body *twice* during each rotation, once on the side PCQ, and again on the side PDQ.

Thus (in England), the sun is due south of C when the meridian of C passes through it at noon. At midnight the sun is again on the meridian, say at D, though then invisible (in England) because it is on the other side of the earth. But if it were visible it would then be due north of C.

The effect of the movement of the meridians, due to the rotation of the earth, is the same (so far as the surveyor is concerned) as if the meridians remained fixed and the heavenly bodies revolved round the earth, each at its proper rate. We shall hereafter frequently speak as if this were actually the case.

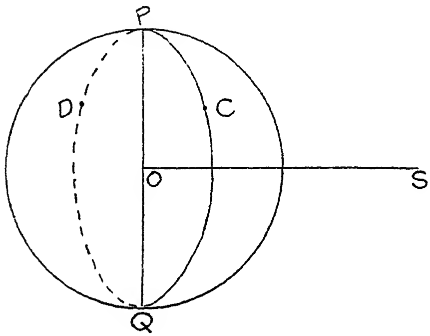


Fig. 64.

Ecliptic.—The plane in which the earth revolves round the sun is called the *ecliptic.*

Strictly the sun and earth both revolve round their common centre of gravity.

The orbit of the earth (*i.e.* the *path* in which it revolves) is elliptical in shape, the sun being at one focus of the ellipse. The earth is *nearest* to the sun on Jan. 2nd (in 1916), and farthest away about July 2nd. These dates vary somewhat after long periods.

Equator.—The plane through the earth's centre perpendicular to the polar axis is called the *equator.* It is practically a fixed plane on the earth. For any one year it may be regarded as fixed also in the heavens, so far as relates to the distant stars, as the earth's axis remains nearly parallel to itself while it revolves round the sun, and the diameter of the orbit is negligible compared to the distance of the stars.

Actually, however, the earth's axis also revolves slowly round a line perpendicular to the ecliptic, completing a cone in about

26,000 years, so that in long periods the equator cannot be regarded as fixed in the heavens.

The polar axis makes a constant angle of about $27\frac{1}{2}°$ with the perpendicular to the ecliptic. It follows that the angle between the planes of the equator and the ecliptic has always this same value. This angle is called the *obliquity of the ecliptic.*

Solar and Sidereal Days.—Now suppose that the line PAS (Fig. 65) represents the meridian plane of a point A on the earth,

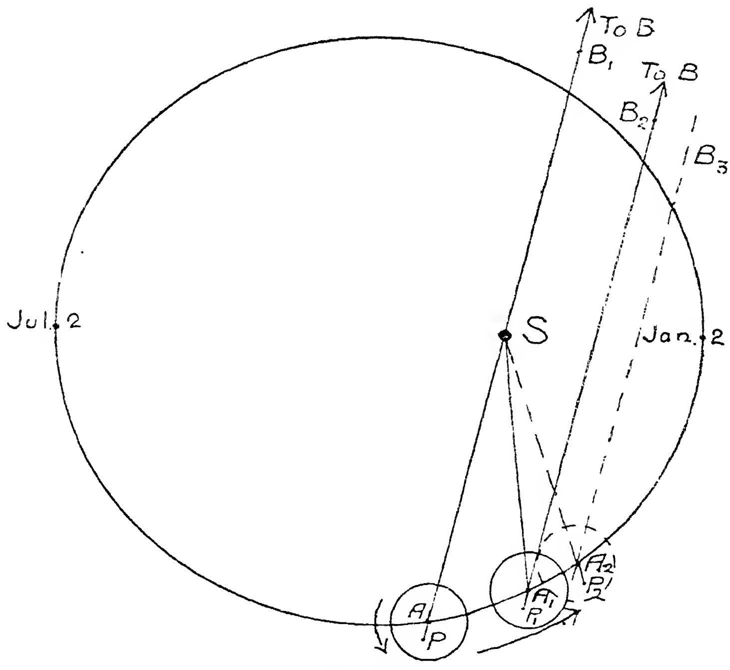


FIG. 65.

at the moment when that plane passes through the centre of the sun S. Farther, suppose that a distant star B is on the meridian at the same instant. Now let the earth make one complete rotation on its axis. The meridian plane of A will then be parallel to its previous direction, as shown at P_1A_1. And as the star B is practically infinitely distant, it will be again on the meridian.

Thus the interval between two successive appearances on any

given meridian of the same distant star is equal to the time which the earth takes to rotate once round its axis.

This interval is called a *sidereal day*, and is a nearly constant interval of time.

Now suppose that whilst the earth is making this rotation, it describes the angle ASA_1 round the sun. Then clearly the meridian P_1A_1 will not pass through the sun at the same time that it passes through B.

In order that it may return to the sun, the earth must describe the farther angle B_3A_2S, which is equal to the angle ASA_2 that it has described round the sun from the first position.

The interval between two successive appearances of the *sun* on the same meridian is called a *solar day*.

Thus the solar day is longer than the sidereal by the time which the earth takes to rotate (round its axis) through the same angle which it describes round the sun in the solar day.

Now in one year of 365·2422 solar days, the total angle described round the sun amounts to 360°. Hence the accumulated extra angle to be described in that time will also be 360°, or one complete rotation. Thus the number of sidereal days in the year is greater by one than the number of solar days. That is, 365·2422 solar days = 366·2422 sidereal days.

Variation in Length of Solar Day.—Now it is well known that the attraction between two bodies like the sun and earth varies at least approximately inversely as the square of the distance between them. Hence the attraction is greater in January than in July. And in order that the centrifugal force due to the earth's revolution may balance this, the earth must move faster round the sun in January than in July. Hence the additional angle to be picked up each day is greater about new year than in midsummer, wherefore the solar day is *longer* at the former time than at the latter, and is *not* a constant interval of time.

(*Note.*—In the diagram the meridian plane is shown always perpendicular to the ecliptic. In consequence of the obliquity of the ecliptic this is not true, and this fact introduces an additional cause of variation in the length of the solar day. The numerical effect of this is no less than that of the cause we have been discussing, but it is less simple, and the student who wishes to study it farther is referred to works on Mathematical Astronomy. It is sufficient for our purpose that he should see clearly that the solar day *does* vary in length.)

Mean Solar Day.—Now, as the sun practically regulates the daily life of mankind, it is desirable that the clocks in ordinary use should follow his movements more or less closely. To do so

exactly, they would have to constantly vary their rate, as we have seen.

Hence astronomers work out the *mean* length of a solar day throughout the whole year. This interval is called a *mean solar day*, and corresponds to 24 hours as shown by an ordinary clock when running perfectly.

Thus suppose that such a clock were set to $12^h\ 0^m\ 0^s$, and started at the moment when the sun was on the meridian of a certain place on a certain day.

Then, next day, when the clock said 12 noon the sun would not as a rule be again on the meridian, but would either have passed it or not have reached it yet, according to the time of year. But on the average of the whole year, it would be sometimes ahead and sometimes behind.

Mean and Apparent Noon.—When the clock, so set, indicates the hour of twelve midday, it is *mean noon* at the place. When the actual sun is on the meridian, it is said to be *apparent noon.*

Mean and Apparent Time.—At any other instant, the number of hours, minutes, and seconds as shown by the clock since *mean noon*, is called the *mean solar time* at that instant and place.

The number of hours, etc., elapsed since *apparent noon* is called the *apparent solar time* at that instant and place.

Greenwich and Local Time.—It is clear, however, that unless two places are exactly north and south of one another, their meridians will be different; hence the sun cannot be on both meridians at once, and the clocks at the two places would have to be started independently, At any instant both the mean time and the apparent time would be different at these two places.

For practical life, it is desirable that the time all over a country of moderate size should be the same. Hence it is usual to choose one standard place, to set the clock to agree with the sun at a certain instant at that place, and then to set all other clocks over the country to agree with that one.

In England the standard meridian chosen is that of the Greenwich Observatory.

Thus the clocks all over the country give what is really the *mean solar time at Greenwich*, and all data in the Nautical Almanac are referred to this time also.

For astronomical purposes, however, we frequently require to make use of the time proper to our actual station, as if our clock were actually set by the sun at that place. Hence we call the time proper to the station *local time*, to distinguish it from the ordinary civil time which is *Greenwich time* really.

We have therefore :—

(*a*) *Greenwich mean time*, which at any instant is the interval (as shown by a perfectly running clock regulated to go 24 hours to the mean length of a solar day) since that clock showed 12 midday. The clock is understood to have been originally set correctly for the meridian of Greenwich.

(*b*) *Local mean time.* This is defined in exactly the same way, except that the clock must be understood to have been originally set for the meridian of the place.

(*c*) *Greenwich apparent time*, which is the interval at any instant since the true sun was last on the meridian of Greenwich.

This interval is understood to be measured in hours, minutes, and seconds, each of such length that if the sun continued to move at the average rate during the interval it would make a complete revolution in 24 of these hours.

(*d*) *Local apparent time*, which is defined in the same way by substituting the words "the place" for "Greenwich."

Mean Summer Time.—The passing of the Daylight Saving Act this year (1916) makes it necessary in England to note also *mean summer time*, which is simply *one hour ahead* of Greenwich mean time, and is in use in ordinary civil life during the summer months.

Equation of Time.—The difference at any instant between the mean and apparent times is called the *equation of time*. It is given, at Greenwich mean and apparent noons for every day in the year, in the Nautical Almanac (pages i. and ii. of each month) and in Whitaker's Almanac.

Its maximum values are about $14\frac{1}{2}$ ms. (clock ahead of sun) in February, and about $16\frac{1}{3}$ ms. (sun ahead of clock) in November. The equation of time is zero *four times* in each year. This arises in consequence of the obliquity of the ecliptic referred to on p. 162, and it is clear that a mean-time clock, if it is set by the actual sun, must be set to agree with it at one of those times.

These occur, as will be seen by referring to the Nautical Almanac, in April and June, about the end of August, and in December.

Mean Sun.—It is convenient to suppose an imaginary sun revolving in the plane of the ecliptic at a uniform rate, in constant agreement with the mean-time clock. This is called the *mean sun*.

Civil and Astronomical Time.—It is occasionally necessary to remember that, whereas the civil day begins at 12 midnight, the astronomical day is supposed to begin at noon on the corresponding civil day, and the hours are reckoned continuously from 0 to 24.

Thus in the Nautical Almanac, June 16th, for instance, extends from noon on June 16th, Greenwich mean time, to noon

on June 17th. Hence 3 a.m. on June 17th civil time is 15^h astronomical time on June 16th, and *vice versâ.*

Abbreviations.—In future we shall use the following abbreviations:—

	G.M.T. =	Greenwich mean time
	G.M.N. =	,, ,, noon
	G.A.T. =	,, apparent time
	G.A.N. =	,, ,, noon
	G.S.T. =	,, sidereal time
also	L.M.T. =	Local mean time
	L.M.N. =	,, ,, noon, and so on, "local" being substituted for "Greenwich."

Examples.—1. Find the G.A.T on December 12, 1916, when the G.M.T. is $7^h\ 20^m$ a.m.

Here the difference is the equation of time.

From the Nautical Almanac, equation of time at G.M.N. on December 12, 1916 = $6^m\ 13{\cdot}8^s$, "to be subtracted from apparent time" (*i.e. added* to mean time). We also see that the equation of time on that day is *decreasing* at the rate of 1·17 sec. per hour.

Now we want the equation of time at $7^h\ 20^m$ a.m. As this is $4\frac{2}{3}$ hours *before* G.M.N. the equation of time will be *greater* than $6^m\ 13{\cdot}8^s$ by $1{\cdot}17 \times 4\frac{2}{3}$.

	h	m	s
Thus equation of time at G.M.N. =		6	13·8
Variation per hour $1{\cdot}17^s$, in $4\frac{2}{3}$h. = +			5·5
Equation of time at $7^h\ 20^m$ . .		6	19·3
G.M.T. =	7	20	0
G.A.T. =	7	26	19·3 a.m.

The corresponding astronomical time is $19^h\ 20^m$ on December 11th.

2. Find the mean summer time at Greenwich on June 26, 1916, when the G.A.T. is $6^h\ 0^m\ 0^s$.

	h	m	s
G.A.T. =	6	0	0
Equation of time at G.A.N. = $2^m\ 34^s$ to be added to apparent time, and increasing $0{\cdot}53^s$ per hour.			
∴ at 6^h after apparent noon =		2	37·2
G.M.T. =	6	2	37·2
∴ mean summer time =	7	2	37·2

The corresponding astronomical time is $6^h\ 2^m\ 37{\cdot}2^s$ on June 26th.

Sidereal Time.—We have seen that the solar day is longer than the sidereal, and hence that a certain distant star which is on the meridian say at midnight to-night, will not be in the same position at the same time by an ordinary clock on future nights.

Hence the mean time on any date does not immediately indicate the position of any distant star with respect to the meridian.

In order that we may have this information, it is desirable that a clock should be made which shall follow the distant stars. Such a clock is called a *sidereal* clock, and registers 24 sidereal hours in one sidereal day.

For setting such a clock some standard star or position in the heavens must be chosen, and the clock set to zero and started at the moment when this star is on the meridian.

First Point of Aries.—The fixed point chosen is not marked by any distant star, but by the position of the sun's centre when crossing the equator at the vernal equinox. It is called the *first point of Aries* because many years ago the sun at this time was in the constellation Aries. The point retains this name, though now some distance from this constellation.

Sidereal Noon and Sidereal Time.—When the first point of Aries is on the meridian of any place, then it is *sidereal noon* at that place, and the sidereal clock should read zero.

Such clocks are used in observatories, and, unlike ordinary clocks, each is set to *local* noon, *not* from Greenwich.

At any other instant the interval of time which has elapsed since sidereal noon is called the *sidereal time* at that instant. It is measured in sidereal hours, minutes, etc.

At the vernal equinox (that is, about March 25th) sidereal noon very nearly coincides with mean noon, as the sun and the first point of Aries are then near together. They separate at the rate of one day in the year, as we have seen, or about two hours per month, so that at the end of April, for instance, sidereal time is about two hours ahead of mean time.

The exact value of the Greenwich sidereal time *at Greenwich mean noon* is given in the Nautical Almanac for every day in the year, on p. ii. of each month.

Relation between Intervals of Mean and Sidereal Time.—We have seen (p. 162) that one year consists of about 365·2422 mean solar days, and 366·2422 sidereal days.

Hence we have the ratio

$$1 \text{ solar day} = 1 \text{ sidereal day} \times \frac{366{\cdot}2422}{365{\cdot}2422}$$

$$1 \text{ sidereal day} = 1 \text{ solar day} \times \frac{365{\cdot}2422}{366{\cdot}2422}$$

The lengths of the hours, minutes, and seconds will be in the same ratio, because the sidereal day, like the solar, is divided into 24 hours.

Tables based on this ratio are given in the Nautical Almanac under the heading "Time Equivalents, Table of," and enable us to

convert intervals of mean solar time to sidereal intervals, or *vice versâ.*

Thus if we want to find the Greenwich sidereal time at 2 p.m. G.M.T. on a certain day, we convert the 2 hours of mean time (= interval since G.M.N.) to a sidereal interval, and add it to the sidereal time *at* G.M.N. on that day.

Example.—Find the Greenwich sidereal time at $7^h\ 25^m\ 6^s$ p.m., G.M.T. on July 6, 1912.

	h	m	s
From Nautical Almanac G.S.T. at G.M.N. on that day =	6	56	23·6
To which we add 7h. =	7	1	9·0
25m. =		25	4·1
6s. =			6·0
∴ G.S.T. =	14	22	42·7

Here $6^h\ 56^m\ 23{\cdot}6^s$ tells the sidereal time when it was noon at Greenwich. It was taken from page ii. of July in the Nautical Almanac.

$7^h\ 25^m\ 6^s$ is the mean time interval since noon. It is converted to a sidereal interval by the conversion tables in Nautical Almanac, and the whole added.

The student should procure a Nautical Almanac of his own and work through the examples. The figures will be slightly different, according to the date of his almanac, but it is useless to attempt to learn astronomy without working examples.

2. Find the Greenwich sidereal time at $4^h\ 25^m\ 6^s$ a.m. on July 6, 1912. Here the mean time is $12^h - 4^h\ 25^m\ 6^s$, or $7^h\ 34^m\ 54^s$ *before* noon. Hence this must be converted to a sidereal interval, and *subtracted* from the G.S.T. at G.M.N.

As the latter is too small for this subtraction to be performed directly, we add 24 hours to it. If we wished to find the time two hours before one o'clock we should add 12 to 1 and subtract 2, giving 11^h, on the same principle. Thus—

	h	m	s
7^h =	7	1	9·0
34^m =		34	5·6
54^s =			54·1
Sidereal interval =	7	36	8·7
G.S.T. at G.M.N. =	6	56	23·6
	24	0	0
	30	56	23·6
As above, sidereal interval =	7	36	8·7
G.S.T. required =	23	20	14·9

3. Find the Greenwich apparent time on October 15, 1912, when the Greenwich sidereal time was $17^h\ 15^m\ 43^s$. Here we must first subtract the G.S.T. at G.M.N. to find the *interval since noon*. This will be a sidereal interval. Convert it to a mean time interval, and it will clearly give the mean time, as the latter starts from mean noon.

Then convert to apparent time as before.

$$
\begin{array}{rr}
 & \text{h} \quad \text{m} \quad \text{s} \\
 & 17\ 15\ 43 \\
\text{G.S.T. at G.M.N.} = & 13\ 34\ 35{\cdot}6 \\
\hline
\text{Sidereal interval} = & 3\ 41\ \ 7{\cdot}4 \\
\hline
3^{\text{h}} = & 2\ 59\ 30{\cdot}5 \\
41^{\text{m}} = & 40\ 53{\cdot}3 \\
7{\cdot}4^{\text{s}} = & 7{\cdot}4 \\
\hline
\therefore \text{G.M.T.} = & 3\ 40\ 31{\cdot}2 \text{ p.m.}
\end{array}
$$

Equation of time at G.M.N. on that day $= 14^{\text{m}}\ 8{\cdot}1^{\text{s}}$, to be added to mean time, and increasing $0{\cdot}55^{\text{s}}$ per hour

$\therefore$ equation of time at $3^{\text{h}}\ 40^{\text{m}}$ p.m. $= 14^{\text{m}}\ 10{\cdot}1^{\text{s}}$, to be added to mean time

$\therefore$ G.A.T. $= 3^{\text{h}}\ 54^{\text{m}}\ 41{\cdot}3^{\text{s}}$

Time and Longitude—We have seen (p. 163) that the true local time at any station is not the same as the Greenwich time, unless the station is on the same meridian as Greenwich.

As the meridians rotate with the earth from west to east, the heavenly bodies appear to revolve from east to west.

A distant star, as we have seen, completes one revolution in one sidereal day, or 360° in 24 sidereal hours. That is, the distant stars appear to move at the rate of 15° per sidereal hour, and if two stations differ in longitude by $n°$, the time taken for any distant star to move from the meridian of one station to that of the other will be $\frac{n}{15}$ sidereal hours.

Conversely, if the time interval be observed as t sidereal hours, then the difference of longitude will be $15t°$.

If the body observed be the sun, the time interval is generally observed by a clock giving mean solar time. Now the true sun in this interval will not, in general, move at the same rate as the mean sun, and hence may describe more or less than 15° per mean solar hour. The observed interval must be corrected for the change in the equation of time during that interval (because that tells how much the true sun has gained or lost on the mean), and then reduced to longitude at the rate of 15° per mean solar hour.

The station whose meridian is crossed last will clearly be *west* of the other station.

If one meridian be that of Greenwich, and the other be a place $n°$ west of Greenwich, it is clear that the mean sun will transit (*i.e.* cross the meridian) at Greenwich $\frac{n}{15}$ mean solar hours *before* it transits at the second station. That is, the mean solar time at Greenwich will be *ahead of the local mean time* by $\frac{n}{15}$ hours.

Similarly, if the place be n° east, the Greenwich time will be *behind* the local by $\frac{n}{15}$ hours.

Also, clearly, the *sidereal times* at Greenwich and the place, at one and the same instant, will differ by $\frac{n}{15}$ sidereal hours (as the first point of Aries moves 15° in 1 sidereal hour), and the Greenwich time will be the smaller if the place be east.

The student will do well to learn the sailors' ungrammatical rhyme :—

"Longitude east, Greenwich least;
Longitude west, Greenwich best."

Example.—(1) On October 17, 1912, a certain star (β Aquarii) was observed to transit at one station at $7^h\ 43^m\ 13{\cdot}6^s$ p.m. G.M.T.

The time of transit on another meridian was $8^h\ 13^m\ 21{\cdot}4^s$ p.m., also by G.M.T. clock. Assuming both clocks right, find the difference of longitude.

$$\begin{array}{lrrr} & \text{h} & \text{m} & \text{s} \\ & 8 & 13 & 21{\cdot}4 \\ & 7 & 43 & 13{\cdot}6 \\ \hline \text{mean time interval} & & 30 & 7{\cdot}8 \\ \hline \left.\begin{array}{l}\text{Reduce to sidereal interval} \\ \text{as body is a distant star}\end{array}\right\} \begin{array}{l} 30^m = \\ 7{\cdot}8^s = \end{array} & & \begin{array}{r}30 \\ \ \end{array} & \begin{array}{r}4{\cdot}9 \\ 7{\cdot}8\end{array} \\ \hline \end{array}$$

$$\text{Longitude in time} = 30\ 12{\cdot}7 = 1812{\cdot}7^s$$

$$\text{Longitude in arc} = 1812{\cdot}7 \times 15'' = 7^\circ\ 33'\ 10{\cdot}5''$$

The second meridian is *west* of the first.

(2) If the sun on December 14, 1912, crossed the meridians of two stations with a mean time interval of $1^h\ 19^m\ 21^s$, find the difference of longitude, given that the equation of time on that day was $5^m\ 17{\cdot}0^s$ to be added to mean time, and decreasing $1{\cdot}2^s$ per hour.

Here it is clear that the mean sun is gaining on the true sun (as the amount to be added to mean time is decreasing), and hence the mean sun would have covered the distance between meridians in *less* time than the true.

Therefore we must *decrease* the time interval.

$$1^h\ 19^m\ 21^s = 1\tfrac{1}{3}^h \text{ nearly.}$$

$$\therefore \text{ change in equation of time} = 1\tfrac{1}{3} \times 1{\cdot}2 = 1{\cdot}6^s$$

$$\therefore \text{ interval for mean sun} = 1^h\ 19^m\ 21^s - 1{\cdot}6^s$$

$$\text{or longitude in time} = 1^h\ 19^m\ 19{\cdot}4^s$$

$$\begin{array}{rl} 1^h = & 15^\circ \\ 19^m = & 4\ 45\ \ 0 \\ 19{\cdot}4^s = & \quad\ 4\ 51 \\ \hline & 19^\circ\ 49'\ 51'' = \text{difference of longitude} \end{array}$$

(3) If the local sidereal time at a station on June 17, 1912, was $16^h\ 8^m\ 13{\cdot}4^s$ when the time by G.M.T. chronometer was $10^h\ 3^m\ 15^s$ p.m., find the longitude.

Here we must first find the Greenwich sidereal time, as the given *local* time is sidereal.

$$\begin{array}{lcr} & & \text{h} \quad \text{m} \quad \text{s} \\ \text{G.S.T. at G.M.N. from N.A.} & = & 5 \;\; 41 \;\; 29{\cdot}0 \\ \text{G.M.T.} = 10^h\,3^m\,15^s \begin{cases} 10^h \\ 3^m \\ 15^s \end{cases} & \begin{array}{c} = \\ = \\ = \end{array} & \begin{array}{r} 10 \;\; 1 \;\; 38{\cdot}6 \\ 3 \;\; 0{\cdot}5 \\ 15{\cdot}0 \end{array} \\ \hline \text{G.S.T. of observation} & = & 15 \;\; 46 \;\; 23{\cdot}1 \end{array}$$

$$\begin{array}{lr} & \text{h} \quad \text{m} \quad \text{s} \\ \text{L.S.T.} = & 16 \;\; 8 \;\; 13{\cdot}4 \\ \text{G.S.T.} = & 15 \;\; 46 \;\; 23{\cdot}1 \\ \hline \end{array}$$

$$\text{Longitude in time} = 21 \;\; 50{\cdot}3 = 1310{\cdot}3^s$$
$$\therefore \text{ Longitude in arc} = 1310{\cdot}3 \times 15'' = 327'\ 34{\cdot}5''$$
$$= 5^\circ\ 27'\ 34{\cdot}5''$$

The longitude is *east*, as the Greenwich time is the smaller (i.e. *least*).

(4) If the sun was on the meridian of a certain station on August 3, 1912, when the G.M.T. was $12^h\ 30^m\ 10{\cdot}3^s$ p.m., find the longitude.

When the sun is on the meridian, local apparent time = zero, or $12^h\ 0^m\ 0^s$.

From this we must find local mean time, to reduce to same units as Greenwich time.

Equation of time at G.M.N. = $6^m\ 0{\cdot}7^s$, to be subtracted from mean time, and decreasing $0{\cdot}20^s$ per hour. Hence at $12^h\ 30^m$ G.M.T. it is $6^m\ 0{\cdot}6^s$. This is to be *added* to apparent time.

$$\begin{array}{lr} & \text{h} \quad \text{m} \quad \text{s} \\ \text{L.A.T.} = & 12 \;\; 0 \;\;\; 0 \\ & 6 \;\; 0{\cdot}6 \\ \hline \text{L.M.T.} = & 12 \;\; 6 \;\; 0{\cdot}6 \\ \text{G.M.T.} = & 12 \;\; 30 \;\; 10{\cdot}3 \\ \hline \end{array}$$

$$\text{Longitude in time} = 24 \;\; 9{\cdot}7 = 1449{\cdot}4^s$$
$$\therefore \text{ Longitude in arc} = 1449{\cdot}7 \times 15'' = 362'\ 25{\cdot}5''$$
$$= 6^\circ\ 2'\ 22{\cdot}5''$$

The longitude is *west*, as the Greenwich time is the greater (i.e. *best*).

(5) Find the G.M.T. at which the sun was on the meridian of a station whose longitude in time was $5^h\ 11^m\ 29{\cdot}5^s$ west on January 15, 1912. Also find the local sidereal time at that station at that instant.

On that day, equation of time at G.A.N. was increasing $0{\cdot}914^s$ per hour. At the succeeding noon, rate was $0{\cdot}887$. Therefore on the average of 5 hours after noon on the 15th we may take rate at $0{\cdot}914 - \dfrac{2{\cdot}5}{24}\,({\cdot}914 - {\cdot}887) = 0{\cdot}91^s$ per hour.

The mean time interval between the stations is $5^h\ 11^m\ 29{\cdot}5^s$, and in this time the equation of time increases $4{\cdot}7^s$.

The equation of time is to be added to apparent time, hence as it is increasing the mean sun is gaining on the true sun.

That is, the true sun will take $4{\cdot}7^s$ longer to describe the journey than the mean sun.

Hence time interval for true sun = $5^h\ 11^m\ 34{\cdot}2^s$.

That is, the transit takes place this amount after G.A.N.

But at G.A.N. the equation of time was $9^m\ 12{\cdot}8^s$, to be added to apparent time.

Hence the G.M.T. was		0	9	12·8
Time interval		5	11	34·2
G.M.T. required =		5	20	47·0
G.S.T. at G.M.N. =		19	34	19·4
G.M.T. reduced to sidereal interval	5^h =	5	0	49·3
	20^m =		20	3·3
	$47{\cdot}0_s$ =			47·1
G.S.T. =		24	55	59·1
Longitude in time =		5	11	29·5
L.S.T. =		19	44	29·6

Celestial Sphere.—As we are not, in surveying, concerned with the distances of the heavenly bodies, we may at any moment suppose them to be set on the surface of a sphere rotating from east to west.

This sphere is called the celestial sphere; its axis is the earth's polar axis prolonged, and the line in which it is intersected by the plane of the earth's equator is called the celestial equator.

Right Ascension.—As the meridian of any station rotates with the earth from west to east, all the heavenly bodies appear to revolve from east to west.

The distant stars all appear to complete one revolution in very nearly the same period (namely, the time the earth takes to make one revolution), as already stated.

Thus these stars all appear to revolve together, keeping the same relative positions with respect to one another, and, of course, with respect to the first point of Aries, which is a nearly fixed point in the heavens.

Hence when the position of the latter is known with respect to any meridian, the positions of all the others should be known too.

For this purpose, it is only necessary to know by how far each star is east or west of the first point of Aries.

These distances have all been measured for the bright stars and are given in the Nautical Almanac under the heading "Apparent Places of Stars." The angular distance of a star measured *from* the meridian of the first point of Aries to the star, always in the direction from *west to east*, right round the circle, and expressed in hours, minutes, and seconds of sidereal time is called the *right ascension* of the star.

Thus the right ascension of a star may have any value between zero and 24 hours. To reduce it to arc, we must allow 15° per hour.

The stars in the Nautical Almanac tables are arranged in order of right ascension.

In consequence of the fact that the earth's polar axis does not

remain quite fixed, and other causes, the right ascensions of the stars do not remain *quite* fixed either, and are therefore given in the table for every tenth day.

Stars near the poles change their positions more quickly than others, in consequence of the movement of the polar axis. Hence these are given in a separate table for every day.

The sun and planets, being nearer to us, change their positions with respect to the distant stars as already shown, and their right ascensions vary much more quickly.

They are given for every day, the sun on p. ii. of each month, and the planets each under its proper name.

The moon being nearest of all, and revolving round us, changes more quickly still in apparent position, and its right ascension is given for every *hour* of Greenwich time on pages v. to xii. of each month.

Hour Angle.—The angular distance of a star from the meridian of a station, expressed in *time* at any instant, is called the *hour angle* of that star at that instant and place.

It is usually understood to be measured *from* the meridian westwards to the star. Thus an hour angle of *one hour* would mean that the star crossed the meridian one hour before the instant considered, and *vice versâ*. An hour angle of 23 hours would mean that it crossed 23 hours before, and hence that it is now *east* of the meridian, and *will cross* again in *one hour* more, as it makes a complete revolution in 24 hours.

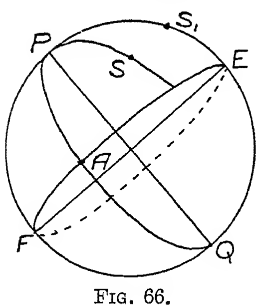


Fig. 66.

The hour angle in such a case is sometimes spoken of as *one hour east.*

It is clear that the sidereal time at any instant tells the hour angle of the first point of Aries, because it tells the time interval since the latter crossed the meridian.

Thus in Fig. 66, let PQ be the polar axis, EAF the celestial equator, PEQ the meridian of the station, and A the first point of Aries. Then, assuming that P is the north celestial pole, the spherical angle EPA, expressed in time at the rate of 15° to 1 hour, is the *sidereal time* at that instant.

Now, if S be any star at the same instant, the angle EPS, expressed in time, is the star's *hour angle.*

But APS, measured backwards from A to the star and expressed in time, is the star's *right ascension*

And clearly EPA = EPS + APS.

Hence we have the rule that at any place and time

Sidereal time = star's hour angle + star's right ascension

As a particular case, consider a star S_1 just crossing the meridian.

EPA = sidereal time.

APS_1 = right ascension of S_1.

But these are equal, as S_1 is in the meridian EP. Hence, *if a star be on the meridian* its right ascension must be equal to the sidereal time.

Its hour angle is then zero, of course, so that the two rules agree.

At 12 noon, mean time, the mean sun is on the meridian. Hence the *sidereal time at mean noon* is equal to the right ascension of the mean sun at that instant, and either term may be used to express it.

Abbreviations.—We shall use the abbreviations R.A. for right ascension, and H. for hour angle.

Examples.—1. If the star Aldebaran was on the meridian of a station on March 20, 1912, when the G.S.T. was $5^h\ 3^m\ 2{\cdot}2^s$, find the longitude.

	h	m	s
From Nautical Almanac, R.A. of Aldebaran =	4	30	51·9
∴ L.S.T. (as star is on meridian) =	4	30	51·9
G.S.T. =	5	3	2·2
Longitude in time =		32	10·3

This is *west*, as the Greenwich time is the greater, and is reduced to arc as before.

2. If the hour angle (in arc) of Algol at a certain station on July 15, 1912, was 76° 46′ east, find the L.S.T.

		h	m	s	
Hour angle reduced to time at 1 hour for 15°	76° =	5	4	0	
	46′ =		3	4	
	H. =	5	7	4	east

As this is *east*, we must subtract it from 24^h.

	h	m	s	
∴ H. =	18	52	56	west
From Nautical Almanac, R.A. =	3	2	27	
∴ L.S.T. =	21	55	23	

3. If the sun's hour angle on May 6, 1912, at a certain station was $4^h\ 15^m\ 21^s$ west when the G.M.T. was $4^h\ 8^m\ 12^s$ p.m., find the L.M.T. and the L.S.T.

The sun's hour angle is the L.A.T.

To reduce this to L.M.T. we must know the equation of time.

Equation of time at G.M.N. = $3^m\ 28{\cdot}5^s$, increasing $0{\cdot}20^s$ per hour, to be subtracted from apparent time.

Therefore at $4^h\ 8^m$ p.m. it is $3^m\ 29{\cdot}3^s$.

$$\begin{array}{rr} & \text{h m s} \\ \text{L.A.T.} = & 4\ 15\ 21 \\ & 3\ 29{\cdot}3 \\ \hline \text{L.M.T.} = & 4\ 11\ 51{\cdot}7 \end{array}$$

To find the L.S.T., remember that any two bodies with the same right ascension must have the same hour angle at the same moment. Hence a star whose R.A. was the same as that of the sun would have the same hour angle, and L.S.T. = hour angle + R.A.

R.A. of sun at G.M.N. = $2^h\ 52^m\ 25{\cdot}0^s$, increasing $9{\cdot}65^s$ per hour.

$$\begin{array}{rr} & \text{h m s} \\ \therefore \text{ at } 4^h\ 8^m\ 12^s \text{ p.m. it was} & 2\ 53\ \ 4{\cdot}9 \\ \text{H.} = & 4\ 15\ 21{\cdot}0 \\ \hline \text{L.S.T} = & 7\ \ 8\ 25{\cdot}9 \end{array}$$

Moon and Planets.—The positions of the moon and planets are usually referred to sidereal time, when it is required to make use of them.

Declination.—The right ascensions of the stars on the celestial sphere correspond with *longitudes* of places on the earth. But just as we must know the latitude as well as the longitude to fix a position on the earth, so in fixing the position of a star we must have a second measurement corresponding to terrestrial latitude.

This is called the star's *declination.*

Thus the declination is the angular distance of the star north or south of the equator.

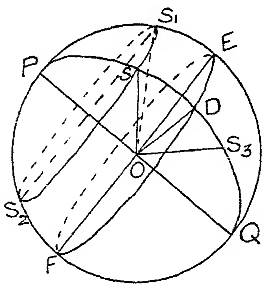


Fig. 67.

The declinations of the sun, moon, planets, and principal stars are tabulated in the Nautical Almanac side by side with the right ascensions.

In Fig. 67, if PQ be the polar axis, EDF the equator, and S a star, DOS is the declination, S_1SS_2 is the path of the star as it appears to revolve round PQ.

PEQF is supposed to be the meridian of the place, and when the star is on the meridian at S_1 its declination is EOS_1, which, of course, is practically equal to DOS for a distant star.

Polar Distance.—The angular distance POS or POS_1 of the star from the elevated pole P is called the *polar distance.*

It is clear that, as the equator is at right angles to the polar axis, the angle POD or POE is 90°. Hence, clearly,

$$\text{Polar distance} = 90^\circ - \text{declination}$$

If the star be on the opposite side of the equator, however, at S_3, DOS_3 = declination, POS_3 = polar distance, and we have

$$\text{Polar distance} = 90° + \text{declination}$$

This is met by calling the declination *minus* in this case, and using the first formula for all cases.

Zenith and Nadir.—The observer's station is necessarily, from his point of view, the top of the earth.

The line through the station and the centre of the earth (assumed spherical) marks the vertical. This line produced meets the celestial sphere in a point Z (Fig. 68) directly overhead called the *zenith*, and a point N directly opposite called the *nadir*.

Then PZQ is the meridian of the place. ZN is the vertical line through the station.

The plane HKI through the centre and at right angles to ZN is the *horizon* plane.

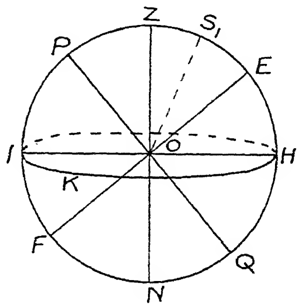


Fig. 68.

Altitude.—The angular distance of a heavenly body above the horizon, at any moment and place, is called the *altitude* of the body.

Thus when the star is on the meridian at S_1, its altitude is HOS_1.

As the altitude constantly varies as the star moves, its value when the star is on the meridian is called the *meridian altitude.*

An inspection of Fig. 67 will make it clear that this has two values, one at S_1, which is the *highest*, and another at S_2, which is the *lowest*, altitude reached by the star.

The star, when at S_1, is said to be at *upper culmination;* when at S_2 it is at *lower culmination.*

Zenith Distance.—The angular distance of the star from the zenith is called the *zenith distance.* Thus for the star at S_1

$$ZOS_1 = \text{zenith distance}$$

The angle ZOH is clearly 90°, hence we have the relation

$$\text{Zenith distance} = 90° - \text{altitude}$$

Latitude.—The latitude of the station, according to the ordinary geographical definition, is the angle between the vertical through the station and the plane of the equator.

Thus in Fig. 68 the angle EOZ is the latitude.

Now $$EOP = 90^\circ$$
$$\therefore EOZ = 90^\circ - ZOP$$

But $$IOZ = 90^\circ$$
$$\therefore IOP = 90^\circ - ZOP$$

Hence $$EOZ = IOP$$

But IOP is the altitude of the pole as measured from the horizon of the station.

Hence we have the rule that *the latitude of any station is equal to the altitude of the pole at that station.*

Co-latitude.—The angle ZOP between the zenith or station and the elevated pole = 90° − latitude, as above. It is therefore called the *co-latitude.*

Abbreviations.—We shall use the following abbreviations :—

h = altitude of star.
δ = declination of star.
z = zenith distance.
p = polar "
ϕ = latitude of station.
l = co-latitude.

Latitude by Meridian Altitudes.—Now in Fig. 68 we have for a star S_1 *on the meridian*

$$EOS_1 = \text{declination} = \delta$$
$$S_1OZ = \text{zenith distance} = z$$
$$EOZ = \text{latitude} = \phi$$

But $$EOZ = S_1OZ + EOS_1$$
$$\therefore \phi = z + \delta$$

This important relation gives the latitude at once in terms of the meridian zenith distance and the declination.

As $z = 90^\circ -$ altitude, it can be found by simply observing, with theodolite or sextant, the altitude of the star when crossing the meridian. The Nautical Almanac tells the value of δ; hence we have at once a means of determining the latitude of the station.

It remains to be seen how far the relation is true for stars which cross the meridian *not* between E and Z.

Case II. The star culminates between E. and H, as S_2 (Fig. 69).

Here $$EOS_2 = \delta$$
$$HOS_2 = \text{altitude} = h$$
$$ZOS_2 = z = 90^\circ - h$$

and clearly $$EOZ = ZOS_2 - EOS_2$$

or $$\phi = z - \delta$$

Here the star is on the opposite side of the equator from the place. That is, if the latitude is *north*, the star S_2 would have *south* declination, and *vice versâ*. That is, we must *subtract* the declination if it be *opposite in name* to the latitude.

Case III. The star culminates between Z and P, as at S_3.

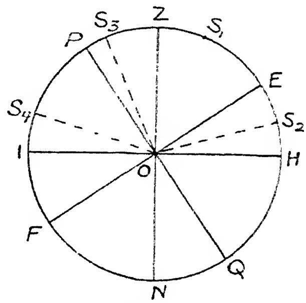


FIG. 69.

Here $EOS_3 = \delta$; $ZOS_3 = z$

and clearly $EOZ = EOS_3 - ZOS_3$

or $\phi = \delta - z$

Thus in this case, where the star is on the *same* side of the zenith as the elevated pole we must take z as *minus*.

Case IV. The star culminates below the pole as S_4. This is a case of lower culmination.

Here $FOS_4 = \delta$
$ZOS_4 = z$

and clearly $EOZ = 180° - (FOS_4 + ZOS_4)$

or $\phi = (180° - \delta) - z$

Thus in this case we again take z as *minus*, and instead of the declination from the Nautical Almanac we take its supplement.

With these conventions, the formula $\phi = z + \delta$ is true for all cases.

Examples.—On January 15, 1912, the corrected meridian altitude of Aldebaran at a station in the northern hemisphere was 46° 18′ 10″.

The star was observed at upper culmination, south of zenith. Find the latitude.

The declination from Nautical Almanac = 16° 20′ 7″ N.

Hence this is Case I.—

$$z = 90° - h = 43°\ 51'\ 50''$$
$$\delta = 16\ \ 20\ \ 7$$
$$\phi = z + \delta = 60\ \ 11\ \ 57\ \text{N.}$$

(2) The star β Ursæ Majoris at a station in the northern hemisphere had an altitude of 16° 41′ 20″ when at lower culmination on October 3, 1912. Find the latitude.

This is Case IV. $\therefore \phi = 180° - \delta - z$.

$$\delta = 56°\ 51'\ 1''\ \text{N.}\quad \therefore 180° - \delta = 123°\ 8'\ 59''$$
$$z = 90° - h = 73\ \ 18\ \ 40$$
$$\phi = 49\ \ 50\ \ 19\ \text{N.}$$

In actual practice a note should always be made of the position of the star.

In examination papers, however, questions are sometimes set without this information, and therefore admitting of more than one answer.

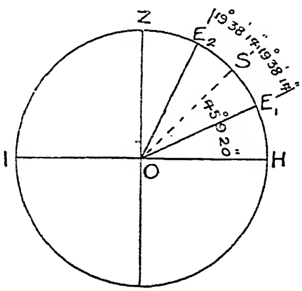


FIG. 70.

3. If the meridian altitude of Arcturus on May 16, 1912, was observed as 45° 9′ 20″, find the latitude of the station. $\delta = 19° 38' 14''$ N.

In these cases it is best to draw a figure. Begin with ZN vertical (Fig. 70), and HI horizontal, as these are always the same. Now show the star at the given altitude above the horizon. It is immaterial on which side of ZN this is done. Thus in the figure HOS$=45° 9' 20''$.

The equator is next marked at the proper distance from the star according to the declination; but as we do not know whether Z is north or south—this information not being given in the question—we do not know on which side of OS to place the equator.

If the latitude were known to have the same name as the star's declination—that is, if it be *north* in this case—Z and S must be on the same side of the equator, and we must place the latter as at E_1, so that $E_1OS = 19° 38' 14''$.

Then this reduces to Case I., and we have

$$\begin{aligned} z &= 44° \ 50' \ 40'' \\ \delta &= \underline{19 \quad 38 \quad 14} \\ \phi &= 64 \quad 28 \quad 54 \text{ N.} \end{aligned}$$

But otherwise it may also be placed at E_2, so that $E_2OS = \delta$.

Here S is known to be *north* of the equator, from the declination. Z falls on the opposite side of E_2, and hence the latitude is south, and we have $\phi = z - \delta$.

whence $\phi = 25° \ 12' \ 26''$ S.

Refraction.—All altitudes, as observed, must be corrected for atmospheric refraction.

The rays of light from a heavenly body traverse the whole depth of the atmosphere, and hence the correction is quite different in amount from that on terrestrial altitudes already described (Chap. III., p. 114).

The amount of this astronomical refraction can be shown to be expressed approximately by a formula $r = \text{A} \tan z + \text{B} \tan^3 z$, where r is the refraction correction, z the star's zenith distance, and A and B are constants to be determined experimentally.

In principle the constants may be determined thus: Suppose that a star is observed directly in the zenith. There $z = 0$, and hence $r = 0$, whatever A and B may be.

The altitude in this position needs no correction, therefore, and the latitude can be determined from such stars.

Then if a star be observed in any other position, we can calculate the true value of z from the known value of ϕ by the formula

$$z = \phi - \delta$$

Comparing this with the observed value of z, the difference gives *one* value for r, corresponding with a known value of z. Two such values would enable A and B to be found.

Actually many values are found and the best values of A and B determined by the method of least squares.

From such observations it has been found that $A = 58$ and $B = 0{\cdot}07$, approximately, whence r is obtained *in seconds*.

Tables based on such formulæ have been drawn up, giving the refraction correction at different altitudes. They are *not* given in the Nautical Almanac, but are to be found in many books of mathematical tables, at the end of this volume, and so on. The constants vary somewhat with the temperature and pressure of the air, and hence are given for mean values of these, the corrections to the mean value for other temperatures and pressures being given in a separate table.

Refraction always makes the body appear *too high*. Hence the refraction correction is to be *subtracted* from altitudes, or added to zenith distances.

Taking the Latitude Observations.—If only an approximate result is required, and the exact direction of the meridian is not known, we choose a star slightly to the east of the meridian as well as we can place it, and set the theodolite on it. It should be slowly rising, and will appear in the theodolite to slowly *fall* below the horizontal hair, assuming that the ordinary inverting eyepiece is being used.

As long as it does so it is followed, and constantly brought back on to the horizontal hair by the tangent screw, until ultimately it will appear to move accurately *along* the horizontal hair (for some minutes), and presently to rise slowly above it. When this happens it is no longer followed, but the altitude is read off. The instrument should be turned horizontally, as the star moves, to keep the latter near the vertical hair, to allow for any slope of the horizontal hair.

If the local sidereal time is known, we can save time by choosing from the Nautical Almanac a star whose right ascension is a few minutes greater than the sidereal time.

The observer must know the constellations to be able to aim

the telescope at the proper star. The student may learn these by the aid of star charts in a little while.

The star culminates when the sidereal time is equal to its right ascension.

When the latitude is approximately known, the zenith distance can be calculated as before by the formula $z = \phi - \delta$.

The theodolite can then be set to the corresponding altitude, and the direction of the meridian being also approximately known, the instrument can be turned into position for observing the star without taking aim by eye. In this way stars too faint to be visible with the naked eye can be used.

The best results for latitude are obtained in this way, using pairs of stars culminating on opposite sides of the zenith, and at as nearly as possible equal distances therefrom.

With such stars, as we have seen, the value of z is positive for one and negative for the other. For equal values of z the refraction correction is equal; hence if there is any small error in the refraction correction (which is always to some extent uncertain), this small error will be the same for both stars and will cancel out on the mean.

The stars are chosen so as to culminate near the zenith, in order that the whole refraction correction may be small; and both stars in any pair must be observed with the *same face* of the instrument, so that collimation errors, etc., may be *the same* for both (in sign as well as magnitude), so that they shall cancel out on the mean of the positive and negative results.

With a good theodolite, using three or four pairs of such stars, latitudes may be found within a second or two of arc.

Zenith Sector and Telescope.—For the best results, the observations are often taken with a *zenith sector*, which is an instrument with a much larger telescope than an ordinary theodolite, and a graduated sector extending to about 15° only on each side of the zenith, instead of a complete vertical circle.

A theodolite capable of giving the same accuracy would be a cumbersome instrument.

In other cases the *zenith telescope*, invented by Captain Talcott of the U.S.A., has been adopted.

These instruments are, however, seldom available for ordinary surveying, and the reader is referred to works such as Clarke's "Geodesy" (The Clarendon Press) for a fuller account of them and of the method of using them.

With these instruments and by the method here described, latitudes can be determined with a probable error of only a small fraction of a second of arc.

Suitable Pairs.—For star south of zenith (case I.)—

$$\phi = z_1 + \delta_1 \text{; or } z_1 = \phi - \delta_1$$

for the second star $\phi = -z_2 + \delta_2$; or $z_2 = \delta_2 - \phi$

As the values of z_1 and z_2 are to be nearly equal, we must have

$$\phi - \delta_1 = \delta_2 - \phi \text{, or } \delta_1 + \delta_2 = 2\phi \text{, very nearly}$$

This gives the necessary relation between the declinations of any pair. The right ascensions should differ by a period no greater than necessary for taking the readings and turning the telescope round, to avoid the risk of any serious change in the refraction coefficients between the observations.

Latitude by Polaris.—As we have seen (p. 176), the latitude of the station is equal to the altitude of the elevated pole.

If there were a star exactly at the celestial pole, it would therefore be merely necessary to observe its altitude, and the latitude would be known.

The pole star (Polaris, or α Ursæ Minoris), though not exactly in this position, is *near* the pole, so that its altitude at any moment needs only a small correction to give the latitude.

The amount and sign of this correction depend upon the position of the star in its path round the true pole, and this depends upon the local sidereal time. Hence the latter must be known for this method.

In Fig. 71, let S be the star, and suppose that its zenith distance has been observed or calculated from the observed altitude.

It is represented by the side ZS in the spherical triangle ZPS.

The side PS = polar distance = 90° − δ, and is known (p. 174).

The angle ZPS is the star's hour angle. This = L.S.T. − R.A. (p. 173), and is therefore known. From these it is required to find the side PZ, which is the co-latitude (p. 176).

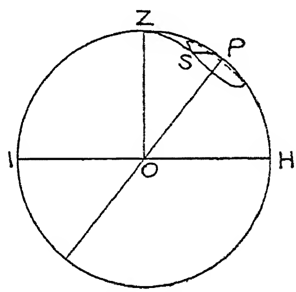


Fig. 71.

We have thus given two sides and the angle opposite to one of them, and we require the third side.

This cannot be directly solved by any of the formulæ in Chap. II., but we may proceed thus:—

$$\cos ZS = \cos PS \,.\, \cos PZ + \sin PS \sin PZ \cos ZPS \text{ (p. 33)}$$

Now put $\cos PS = k \cos A$

and $\sin PS \,.\, \cos ZPS = k \sin A$

We can find k and A from these equations, for by division

$$\tan A = \tan PS . \cos ZPS$$

and k can then be found by either equation.

Then, putting $ZS = z$, we have

$$\begin{aligned} \cos z &= k \cos A \cos PZ + k \sin A \sin PZ \\ &= k \cos (PZ - A) \end{aligned}$$

Hence PZ − A can be found, and therefore PZ as A is known. This formula is not very convenient, however, for use with small angles.

It is more usual to write $PZ = ZS + x$, where x is the correction to be added to the observed zenith distance to find the co-latitude.

The formula is then arranged to find x, and this, being a small angle, can be expanded in terms of its trigonometrical functions, so that we obtain the value of x directly instead of its sine or cosine.

The Nautical Almanac gives tables for determining the correction at once (in terms of the sidereal time, date, and altitude) based on such a formula. The tables will be found under the heading "Pole Star, Tables to find latitude by the."

It is to be observed that before using these tables a constant correction of *one minute* must be subtracted from the observed altitude (*vide* example in the "Explanation" in N.A.).

Taking the Observations.—The advantage of this method is that the observations can be taken at any time. There is no need to wait for suitable pairs of stars. The disadvantage is that we cannot eliminate any errors in the refraction correction. Altitudes should be read alternately face left and face right, and the mean value taken for each pair of readings, to give one determination of latitude.

The readings should not be separated by more than about two minutes of time.

The time may be read either directly on a sidereal clock suitably placed, or on a mean-time watch. If the latter, it is allowed to lie on the horizontal plate of the theodolite, and the observer looks immediately at the second hand as soon as he has brought the hairs to coincide with the star.

The *best* times are within an hour or two of upper or lower culmination.

Thus, say we were observing in 1912, at the end of July. This is about four months from the vernal equinox, hence the sidereal time is about eight hours ahead of mean time (p. 166).

Now the R.A. of polaris was about $1\frac{1}{2}$ hours.

Hence L.S.T. of upper culmination = $1^h\ 30^m$ about
and L.S.T. of lower culmination = $13^h\ 30^m$

The L.M. times corresponding are about 8 hours behind these, or $17^h\ 30^m$ (*i.e.* $5^h\ 30^m$ a.m.) and $5^h\ 30^m$ p.m. respectively. The second would be two long before dark, and it would be best to read just before daylight.

Necessary Accuracy for Time.—At the worst, Polaris may change 5″ in altitude in about 16 seconds of time.

If it be not more than 2 hours from culmination, about 5″ in 32 seconds.

Hence the local sidereal time must be pretty accurately known if the error is not to exceed that likely to occur in a single reading with a good 5-inch theodolite.

This is the greatest drawback to the method, but if the time *is* accurately known, it is quite easy to get a result not more than 2 or 3 seconds from the correct value, in about 4 readings with a theodolite reading to 10″.

Barometer and thermometer should of course be observed to get the refraction correction.

Example.—On July 17, 1912, the altitude of Polaris was observed as 52° 18′ 22·5″. Barometer, 29·87″; thermometer, 56° F.; L.S.T. = $22^h\ 46^m\ 13^s$. The result is corrected for instrumental errors, but not for refraction.

From refraction tables (Bremiker's Mathematical Tables), mean refraction = 44·6; barom. correction = +0·4; thermometer correction = +0·8; ∴ total = 45·8″.

	°	′	″
∴ observed altitude =	52	18	22·5
Refraction			−45·8
Corrected altitude =	52	17	36·7
z =	37	42	23·3

(*a*) Working by formula—

	h	m	s
L.S.T. =	22	46	13
From Nautical Almanac, R.A. =	1	27	39·5
Hour angle =	21	18	33·5
That is,	2	41	26·5 *east*

To find ZPS we reduce this to arc at 15° per hour.

	°	′	″
2^h =	30	0	0
41^m =	10	15	0
$26 \cdot 5^s$ =		6	37
ZPS =	40	21	37

$$\text{From Nautical Almanac, } \delta = 88^\circ\ 50'\ \ 2{\cdot}5''$$
$$\therefore \text{PS} = 1\ \ \ 9\ 57{\cdot}5$$
$$\therefore \tan \text{A} = \tan 1^\circ\ 9'\ 57{\cdot}5'' . \cos 40^\circ\ 21'\ 37''$$
$$\text{whence A} = 53'\ 18{\cdot}6''$$
$$k = \frac{\cos \text{PS}}{\cos \text{A}} \quad \therefore \log k = \bar{1}{\cdot}9999623$$
$$\cos(\text{PZ} - \text{A}) = \frac{\cos z}{k} = \frac{\cos 37^\circ\ 42'\ 23{\cdot}3''}{k}$$

$$\begin{array}{rrrrl} & {}^\circ & {}' & {}'' & \\ \therefore \text{PZ} - \text{A} = & 37 & 42 & 0 & \text{to the nearest second} \\ \text{A} & & 53 & 18{\cdot}6 & \\ \hline \text{PZ} = & 38 & 35 & 18{\cdot}6 & \\ \therefore \phi = & 51 & 24 & 41{\cdot}4 & \text{N.} \end{array}$$

(*b*) Working by Nautical Almanac—

$$\begin{array}{lrrr} & {}^\circ & {}' & {}'' \\ \text{Corrected altitude} = & 52 & 17 & 36{\cdot}7 \\ \text{Constant correction} & & 1 & 0 \\ \hline & 52 & 16 & 36{\cdot}7 \\ \text{From Nautical Almanac, 1st correction} & & -53 & 27{\cdot}0 \\ \hline & 51 & 23 & 9{\cdot}7 \\ \text{,, \quad ,, \quad 2nd \quad ,,} & & & +23{\cdot}1 \\ \text{,, \quad ,, \quad 3rd \quad ,,} & & +1 & 9{\cdot}7 \\ \hline \phi = & 51 & 24 & 42{\cdot}5 \end{array}$$

The disagreement of one second is probably due to the fact that the various corrections in the Nautical Almanac are given to the nearest second only, but one can scarcely hope for a result nearer than this under ordinary conditions by this method.

Latitude by Circum-meridian Altitudes of the Sun.—When it is desired to avoid night work we can find the latitude of the station by observing a single meridian altitude of the sun, according to the method on p. 176. But as we have no choice of suns to arrange suitable pairs, as with the distant stars, we should be compelled to rely on *one* observation. Thus it would be impossible to check the result or to eliminate the instrumental errors by reversing face, or to obtain greater accuracy than could be given by one reading.

Now if an observation be taken *near* the meridian, and the sun's hour angle be known, it is possible to calculate the small correction to be added to the observed altitude to find what the corresponding *meridian* altitude would be. We can therefore take several altitudes (extending say from ten minutes before apparent noon to ten minutes after noon), face left and face right alternately, reduce each observation to the meridian as above, and calculate the latitude from the *mean* value of the meridian altitude. This is the method of *circum-meridian altitudes.*

In Fig. 72, if S be the sun we have

$$\cos ZS = \cos ZP \,.\, \cos PS + \sin ZP \,.\, \sin PS \,.\, \cos ZPS$$

But, as before,

$$ZP = 90^\circ - \phi$$
$$PS = 90^\circ - \delta$$

and put $\quad ZPS = H = \text{hour angle}$

$$\therefore \cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \,.\, \cos H$$

But $\quad \cos H = 1 - 2 \sin^2 \frac{H}{2}$

$$\therefore \cos z = \sin \phi \sin \delta + \cos \phi \cos \delta - 2 \sin^2 \frac{H}{2} \cos \phi \cos$$
$$= \cos (\phi - \delta) - 2 \sin^2 \frac{H}{2} \,.\, \cos \phi \cos \delta$$

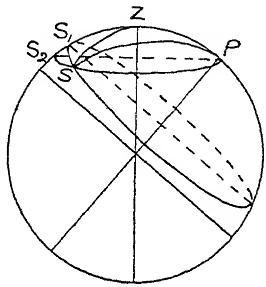


FIG. 72.

Now if S_1 be the position of the sun when crossing the meridian, and we put

$$ZS_1 = z_0 = \textit{meridian}\text{ zenith distance}$$

then $\quad \phi - \delta = z_0$, as on p. 179

$$\therefore \cos z = \cos z_0 - 2 \sin^2 \frac{H}{2} \cos \phi \cos \delta$$

or $$\cos z_0 - \cos z = 2 \sin^2 \frac{H}{2} \cos \phi \cos \delta$$

that is $$2 \sin \frac{z + z_0}{2} \,.\, \sin \frac{z - z_0}{2} = 2 \sin^2 \frac{H}{2} \cos \phi \cos \delta$$

Now $z - z_0$ is the small difference between the observed and meridian altitudes, which we desire to calculate. It is represented (though exaggerated) by S_1S_2 in the figure.

For the purpose of calculating it we may put $\frac{z + z_0}{2} = z_0$, as z and z_0 are nearly equal.

If we thereby introduce an error of say 1 part in 1000 in the value of $\sin \frac{z + z_0}{2}$, this will cause a similar percentage error in the result. But the amount of this will be negligible, as the whole correction is small.

$$\therefore \sin \frac{z - z_0}{2} = \frac{\sin^2 \frac{H}{2} \cos \phi \cos \delta}{\sin z_0}$$

Now, for small angles the sine is very nearly proportional to the angle. Hence if x be the value of such an angle in seconds, we have

$$\sin x : \sin 1'' :: x : 1$$

or

$$\sin x = x \times \sin 1''$$

$$\therefore \sin \tfrac{1}{2}(z - z_0) = \tfrac{1}{2}(z - z_0) \times \sin 1''$$

whence, by substitution in the above equation,

$$\tfrac{1}{2}(z - z_0) \times \sin 1'' = \frac{\sin^2 \frac{H}{2} \cos \phi \cos \delta}{\sin z_0}$$

or

$$z - z_0 = \frac{2 \sin^2 \frac{H}{2} \cos \phi \cos \delta}{\sin z_0 \sin 1''}$$

This gives the value of the correction in seconds, to be added to the observed *altitude* or subtracted from the observed zenith distance.

To reduce it to figures, we first take the *highest* observed altitude and subtract from 90° to find a value of z_0.

Find from this an approximate value of ϕ by the formula

$$\phi = z_0 + \delta$$

These values of ϕ and z_0 will be sufficiently close for calculating the correction, for the reason already given, though a slightly better result is obtained if the *mean* observed zenith distance be used for z_0 in the denominator.

The time must be taken for each observation, and the corresponding value of H determined as below.

Then work out the value of $\dfrac{2 \sin^2 \frac{H}{2}}{\sin 1''}$ for each, *take the mean*, and multiply by $\dfrac{\cos \phi \cos \delta}{\sin z^0}$.

This gives the *mean value* of the correction, and is to be added to the *mean of the observed altitudes* to get the mean value of the meridian altitude.

To reduce the labour, values of $\dfrac{2 \sin^2 \frac{1}{2}H}{\sin 1''}$ are often tabulated. Such a table will be found at the end of this book.

To find H, we require to know the time interval between the observation and the instant of the sun's transit, or local apparent noon. This interval should strictly be expressed neither in mean time nor sidereal, but according to the rate of motion of the true

sun. It is seldom, however, that the equation of time changes appreciably during the observations, and it is sufficient to measure the intervals by a mean-time clock.

The time of local apparent noon by this clock is found in different ways, according to circumstances.

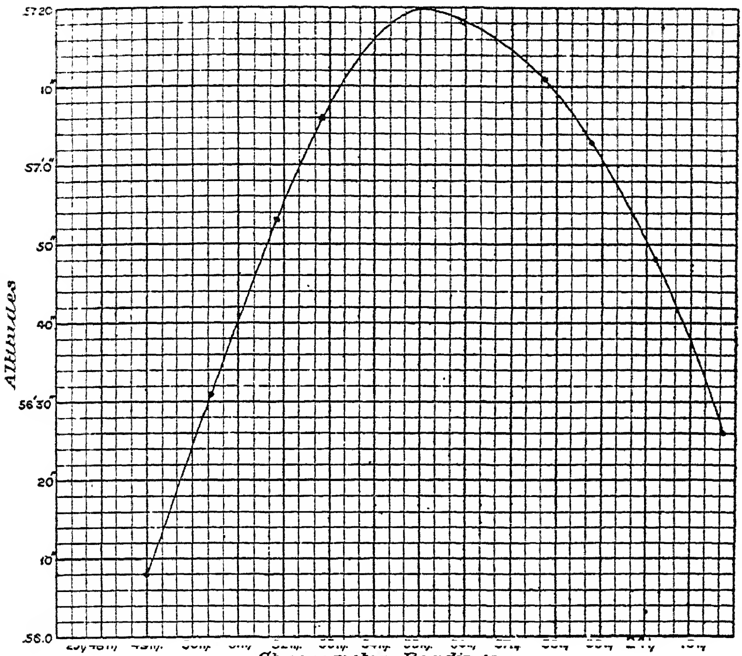


Fig. 73.

If the clock gives local mean time and its error be known, and the longitude is also approximately known, we can find the G.M.T., and hence the equation of time, as in previous examples. This equation of time is to be added to or subtracted from the clock time of each observation to find local apparent time, which gives the hour angle at once.

If local time is not accurately known, separate observations may be made for this, as described later. Or the instant of apparent noon may be found graphically by plotting the observations in a curve, as in Fig. 73.

The observations there shown were a poor set, but a mean curve should be drawn as well as possible to give a shape like a parabola, and, before plotting, all altitudes should be corrected for index errors of the theodolite (if a theodolite is used), which may be found as on p. 249, Part I. It is to be remembered, however, that the index error nearly always varies with the focus (*vide* Part I., p. 249, for reason), and must therefore be found on a very distant object.

Then if any horizontal line be drawn to cut the curve, the time corresponding to the middle of this chord gives the instant of apparent noon nearly.

In the figure, take the line corresponding to altitude 57′. The left end cuts the curve at time $52^m\ 22^s$, the right end at $59^m\ 4^s$. Hence the summit is at the mean, or $55^m\ 43^s$.

The means for the chords at 57′ 10″ and 56′ 50″ are $55^m\ 32^s$ and $55^m\ 49^s$ respectively. For the mean take $23^h\ 55^m\ 41^s$.

The hour angle for each observation is the difference between this and each observed time.

It is reduced to arc at the rate of 15 seconds of angle for each second of time.

If the body observed is a star, the hour angles must be expressed as sidereal intervals.

Hence, if read on a mean-time clock they must be reduced as previously described.

In either case, if the clock has a rate (gaining or losing), the hour angles should be corrected accordingly.

Circum-meridian altitudes of stars may be observed where latitude is to be found from one or two bright stars only.

The method should never be used with a body culminating less than 20° from the zenith, and the greatest hour angle must not exceed 20 minutes of time.

We have hitherto supposed that the declination remains constant during the observations. This is practically true for a star, but in the case of the sun the declination used in all the formulæ should be that corresponding to the time of local apparent noon, as well as it can be ascertained. And if $d\delta$ be the rate of *increase* of the sun's declination in *seconds per minute*, E the sum of the east hour angles, and W the sum of the west hour angles, both in minutes, and n the number of observations, the mean of the observed altitudes must be corrected by a farther amount $\frac{(E-W)d\delta}{n}$ seconds. The sign of this depends on the sign of $d\delta$ and the magnitudes of E and W.

Near the meridian an increase of declination causes an equal

increase of altitude. Hence, if the declination is increasing, each altitude on the east (i.e. *before* noon) will be smaller than it would have been if the declination had been throughout the same as at noon. Conversely, all those on the west will be too high. The above formula clearly follows. Alternatively we may correct each altitude separately. If H = hour angle, correction = $H \times d\delta$, to be added algebraically to *east* observations and subtracted from west.

Example.—We shall first take an example of a star.

In longitude $5^h\ 5^m\ 8^s$ E., and approximate latitude 51° 20′ N., on September 1, 1912, Altair was observed with a sextant (index correction, −24′ 50″) and mean-time chronometer ($1^m\ 23^s$ fast). Barometer, 30·07 ins.; thermometer, 60° F.

Chron. time.	Double altitude.
h m s	° ′ ″
9 1 17	94 49 0
9 2 54	49 40
9 4 36	49 50
9 7 0	50 0
9 9 11	49 20
9 12 0	48 40
9 14 30	48 10
	7)344 40
Mean =	94 49 14·3

Here the first step is to find the chronometer time of transit. In this case the data are given for calculating this.

	h m s
From Nautical Almanac, R.A. of Altair =	19 46 32
∴ L.S.T. of transit =	19 46 32
Longitude correction =	−5 5 8
G.S.T. of transit =	14 41 24
G.S.T. at G.M.N. =	10 41 7
Sidereal interval since G.M.N =	4 0 17
4^h =	3 59 20·7
17^s =	17·0
Mean time interval = G.M.T. of transit =	3 59 38
Longitude	+5 5 8
L.M.T. of transit =	9 4 46
Chronometer correction	+1 23
Chronometer time of transit =	9 6 9

Next take the difference between this and each observed time. The results will give the hour angles in mean time intervals. These are reduced to sidereal intervals, and then to arc at 15″ to 1^s. The results give the values of H. Then calculate the values of $\frac{2 \sin^2 \frac{1}{2}H}{\sin 1''}$ (or these may be read

off from the table, see p. 186), and find the mean. In the table below $T_0 = 9^h\ 6^m\ 9^s$: T = chronometer time for each transit, and $m = \frac{2 \sin^2 \frac{1}{2}H}{\sin 1''}$.

No.	$\pm T_0 - T$.	Sidereal interval	Ditto in arc = H.	½H.	m.
	m s	m s	° ′ ″	° ′ ″	
1	4 52	4 53	1 13 15	36 37·5	46·8
2	3 15	3 16	49 0	24 30	20·9
3	1 33	1 33	23 15	11 37·5	4·7
4	51	51	12 45	6 22·5	1·4
5	3 2	3 2	45 30	22 45	18·1
6	5 51	5 52	1 28 0	44 0	67·6
7	8 21	8 22	2 5 30	1 2 45	137·4
					7)296·9
				Mean =	42·4

The next step is to find the approximate values of ϕ and Z_0.

Mean refraction = 53·8″
Barometer = +0·1
Thermometer = −1·1
Refraction = 52·8

	° ′ ″
Highest observed double altitude =	94 50 0
Index error =	−24 50
	94 25 10
Altitude =	47 12 35
Refraction	53
Corrected altitude =	47 11 42
Approximate z_0 =	42 48 18
From Nautical Almanac, δ =	8 38 13 N.
Approximately, ϕ =	51 26 31

The mean value of m in the table above is 42·4. Hence mean value of correction is $42{\cdot}4 \times \frac{\cos \phi \cos \delta}{\sin z_0}$. This gives 38·5″, about.

	° ′ ″
Mean double altitude =	94 49 14·3
Index error =	−24 50
	94 24 24·3
Mean observed altitude =	47 12 12·1
Refraction	52·8
Mean altitude =	47 11 19·3
Correction as above	38·5
Meridian altitude =	47 11 57·8
True z_0 =	42 48 2·2
δ =	8 38 13·4
ϕ =	51 26 15·6 N.

Sun Observations

The main points of difference between observations of the sun and those of the fixed stars are—

1. That whereas the right ascensions and declinations of the fixed stars are practically constant—and hence do not change during twenty-four hours, or during the time occupied by a set of observations—those of the sun, moon, and planets may change appreciably during the same interval.

Hence, if an observation of the sun be made, it is necessary to find the corresponding *Greenwich time*, in order that the declination of the sun at that instant may be obtained from the Nautical Almanac for purposes of computation.

The maximum rate of change of declination—that is, at the vernal and autumnal equinoxes—is, however, only about one minute per hour, so that, if a G.M.T. chronometer is used, it is only necessary that its error should not exceed some two or three minutes, in order that the declination may be known with an error not exceeding the probable error of observation with an ordinary instrument. Similarly, if the clock gives L.M.T., the longitude need not be known with great accuracy.

At the solstices—midsummer and midwinter—the rate of change is much slower, and a larger error in the Greenwich time makes no appreciable difference.

The movements of the moon are, however, much more rapid. The declination of the moon may change as much as fifteen minutes per hour, and consequently the data for computing Greenwich time must be known with considerable accuracy to make lunar observations of any value.

The declination being known, the same formulæ can be applied to the sun as to the fixed stars.

Inversion of Image.—Secondly, a fixed star being for practical purposes a mere point, it can be exactly bisected by the cross hairs, and no complication is introduced by the inversion of the image, which takes place in every inverting telescope, such as that of the ordinary theodolite.

But the sun's disc being of very appreciable magnitude—rather more than half a degree—it is impossible to bisect it with certainty, and hence the observations are taken to the edges of the disc, as shown in Fig. 74, in which KL and MN represent the cross hairs and ADEB the sun's image, as seen through the telescope.

It is usually necessary to get both the vertical and horizontal hairs tangent to the disc as shown.

The Nautical Almanac gives the angular value of the sun's

semi-diameter—which varies with the distance of the sun from the earth—for every day of the year. Thus, if the altitude of the edge A be read off, that of the centre C can be found by applying the correction for semi-diameter.

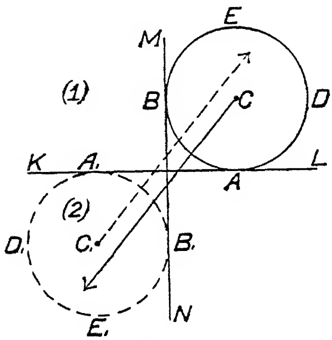


FIG. 74.

The inversion of the image by the telescope must now be taken into account, however. With the position of the image shown the correction is to be *subtracted*, inasmuch as the inversion in a vertical direction takes place about the plane KL, and hence the real disc is *below* that plane, and the point A, which is the *lower* edge or limb of the image, is the *upper* limb of the true sun.

Similar reasoning applies to horizontal angles, and the true position of the sun to give the image ADEB would be that shown by the dotted circle. Hence, to get the azimuth of the sun's centre from that of the edge B with a theodolite graduated from left to right, as usual, the correction for semi-diameter must be *subtracted*, because we appear to be observing the *left edge* B of the sun; hence we are *really* observing the *right edge* as at B_1, and therefore the horizontal circle is giving a *greater* reading than it would at the centre.

The *magnitude* of the correction is equal to the given semi-diameter in the case of vertical angles. But for horizontal angles this value must be multiplied by sec α, if α be the altitude.

The reason of this will be clear by reference to Fig. 19, p. 37, where we may regard P as the zenith and AB as the semi-diameter.

The Nautical Almanac gives this as observed at 0. Now the correction to the horizontal angle is the angle BPA in the triangle BPA, which is right-angled at A. And, solving,

$$\begin{aligned}\tan P &= AB \,.\, \operatorname{cosec} AP \\ &= \tan AB \,.\, \sec \text{altitude}\end{aligned}$$

As P and AB are both small, we can write the actual angle for each instead of the tangent, whence the above formula.

If the diagonal eyepiece (Part I. p. 66) be used, the image is seen without inversion in a *vertical* direction, but is still inverted horizontally, so that the top looks like the top, but right looks like left, and *vice versâ*.

Recording Observations.—It is evident that some regular system of recording such observations must be followed to avoid misunderstandings as to which limb was actually observed. In the Field Book, they may be marked to be recorded as seen through the telescope. This may tend to prevent errors in the field, particularly with beginners.

It is perhaps more usual and more convenient, however, to record the readings as referring to the true sun. Thus, in the figure, the altitude of the point A would be recorded as that of the sun's *upper* limb, and the azimuth of B as that of the *right* limb.

Observing both Limbs.—It is more satisfactory to observe both limbs, so as to eliminate the correction for semi-diameter.

In the northern hemisphere, outside the tropics, we are usually looking in a southerly direction when observing the sun.

Hence the east, whence the sun is rising, will be on the left, and if we watch him through the telescope when rising he will appear to move somewhat in the direction shown by the full arrow in Fig. 74, his true path being that shown by the dotted arrow. If setting in the northern hemisphere, he will appear to move upwards from right to left, assuming that we are using the ordinary inverting eyepiece.

To observe both limbs, then, it is convenient, having made contact with both hairs in one quadrant, to allow one hair to remain fixed until the sun, by his own movement, makes contact with it on the other side of it, following him with the second wire only.

Thus, suppose the sun rising in the northern hemisphere, and that we decide to let the horizontal hair remain fixed.

Make contact with both hairs, as at AB, Fig. 75 (*a*), the true sun being below the hair. Read the time and also the horizontal limb of the theodolite, to obtain the azimuth of the sun's right limb.

The altitude circle need not be read at this stage, as it will remain fixed.

By the time the readings are taken the image will be about in the position shown in Fig. 75 (*b*), and as the image moves it is clear that the point E will ultimately make contact with the wire KL on the lower side.

But in order that the point D may make contact with MN at the same instant, the latter must be moved.

If we move it towards D as seen in the image, the latter will move further away from it. It must be moved in the opposite direction until the hair and image occupy some such position as shown by the dotted lines.

Then carefully follow the disc with the hair MN by means of the horizontal tangent screw, keeping the hair always so near the edge that it is just visible, until the moment when contact is made with the horizontal wire. At that moment the vertical wire should be made tangential also (Fig. 75, *c*).

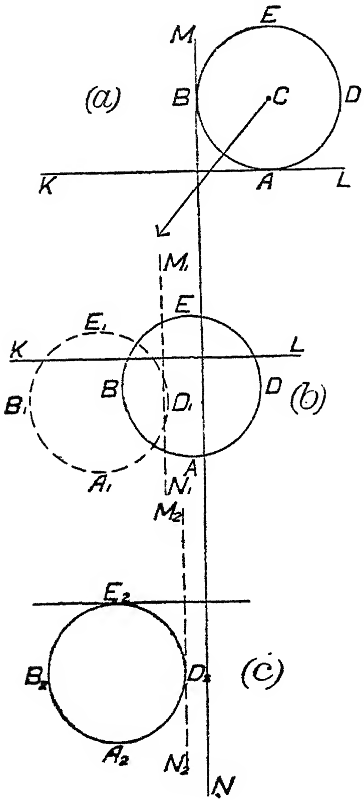


FIG. 75.

The time and azimuth are again read, and also the altitude circle if it has not been read previously.

These observations should be repeated with reversed face, and the means of the four readings—one pair with face left and one with face right—will give the altitude and azimuth of the sun's centre and the time to be used in the calculations.

This eliminates the semi-diameter correction, as well as index error.

Some surveyors prefer to reverse face immediately after taking the sun in one quadrant, and to observe it in the opposite quadrant with reversed face. In this case the *two* observations will make one complete set, free from index error and semi-diameter corrections.

In the particular case of circum-meridian altitudes, these suggestions must be modified by the fact that horizontal angles are not required, and the sun is moving very slowly in altitude.

It is therefore only necessary to bring the horizontal hair tangent to the upper and lower edges (or *limbs* as they are called) alternately, while keeping the sun about in the middle of the field horizontally.

It is, moreover, perhaps wiser in this case *not* to take the means of two or four observations, but to correct each observation separately.

Parallax.—The third difference between sun and star observations arises from the comparative proximity of the sun to the earth.

In consequence of this there is an appreciable difference between the lines drawn from the sun's centre to the centre of the earth and to a point of observation on its surface.

This difference clearly diminishes as the sun approaches the zenith, where the two lines will coincide.

This is shown in Fig. 76, where E is supposed to represent the earth, and S the sun.

A being the point of observation, the true celestial horizon is OM, and the true altitude of the sun is MOC, or a_1. But the natural horizon as given by the theodolite is AN, and the measured altitude is NAC, or a_2, which differs from a_1 by the angle ACO, or β. When the sun is near the zenith Z this approaches zero, as shown by the dotted lines.

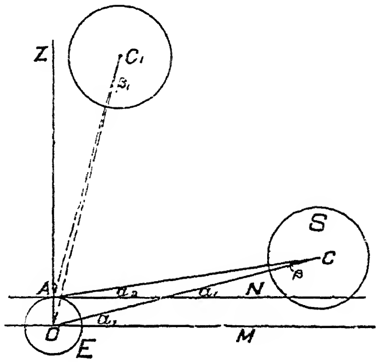


FIG. 76.

The correction for parallax reduces the observations made on the surface to the centre of the earth. The value of the correction is given in Appendix B, at the end of this book.

It is clear that when the sun is on the horizon the parallax ACO has its maximum value.

$$\text{In that position } \sin \text{ACO} = \frac{\text{AO}}{\text{CO}} = \frac{\text{earth's radius}}{\text{distance, earth to sun}}.$$

But ACO in seconds : 1 second : : sin ACO : sin 1″,

$$\therefore \quad \text{ACO} = \frac{\text{earth's radius}}{\text{distance CO} \times \sin 1''}$$

This is called the horizontal parallax, and its value is usually rather less than 9″ (see under altitude 0°, Appendix B).

It varies with the sun's distance from the earth, and is given in the Nautical Almanac for every day of the year on page i. of each month.

For any other altitude, a,

$$\text{AC}_1\text{O} = \frac{\text{AO} \cos a}{\text{C}_1\text{O}} = \text{horizontal parallax} \times \cos a$$

Level Error.—In observing altitudes of any heavenly body with a theodolite, the level cannot be corrected for each reading (inasmuch as the body is moving), hence we must read it at each observation.

The method of reading and applying the correction is given on p. 15.

Example on the Sun.—Take the following observations near Cheltenham on July 18, 1916 :—

Reading of vertical circle.	Level.		Face.	Limb.
	O.	E.		
° ′ ″				
59 19 55	5	9·5	L.	T.
58 48 55	5	10	L.	B.
58 53 10	11	4	R.	B.
59 26 25	6·5	8	R.	T.
59 26 20	6·5	8·5	R.	T.
58 55 20	5·5	9·5	R.	B.
58 48 55	7·5	7·5	L.	B.
59 19 50	8·5	6·5	L.	T.

Face left readings are marked L.; face right, R. Readings to the true top of the sun are marked T.; those to the bottom, B.

The index error (Part I. p. 247) was 2′ 37″, to be *added with left face*; the sun's semi-diameter from Nautical Almanac was 15′ 46″. This correction, as already shown, must be *added* when the bottom of the sun is observed, and subtracted when the top is observed.

The declination at G.M.N. was 21° 3′ 7″ N., decreasing 26″ per hour.

The value of one division of the bubble was 20″. Thus for the first observation level correction $= \frac{O - E}{2} \times 20'' = \frac{5 - 9\cdot5}{2} \times 20'' = -45'$.

Working in this way we obtain the following results :—

Level correction.	Index error.	Semi-diameter.	Total correction.	Corrected altitude.	Time by watch.
″	′ ″	′ ″	′ ″	° ′ ″	h m s
− 45	+ 2 37	− 15 46	− 13 54	59 6 1	1 0 3
− 50	+ 2 37	+ 15 46	+ 17 33	59 6 28	1 58
+ 70	− 2 37	+ 15 46	+ 14 20	59 7 30	3 52
− 15	− 2 37	− 15 46	− 18 38	59 7 47	6 55
− 20	− 2 37	− 15 46	− 18 43	59 7 37	7 55
− 40	− 2 37	+ 15 46	+ 12 49	59 7 49	9 2
0	+ 2 37	+ 15 46	+ 18 23	59 7 18	11 9
+ 20	+ 2 37	− 15 46	− 12 49	59 7 1	12 32
			Mean =	59 7 11·4	

The watch was known to be about six minutes slow on mean summer time (= 54^m fast on G.M.T.), and we shall assume the longitude unknown with any accuracy.

By plotting, the watch time of transit was found to be about $1^h\ 7^m\ 36^s$.

H = hour angle.	$m = \dfrac{2\sin^2 \frac{1}{2}H}{\sin 1''}$.
m s	
7 33	111·9
3 38	62·3
3 44	27·4
41	0 9
19	0·2
1 26	4·0
3 33	24·7
4 56	47·8
Mean =	34·90

	°	′	″
Highest observed altitude =	59	7	49
Refraction =			−33·5
	59	7	15·5
(9 × cos 59° 7′) parallax =			+4·5
Corrected altitude =	59	7	20
Approximately, z =	30	52	40

	h	m	s
Watch time of transit =	1	7	36
Watch fast		54	0
G.M.T. =		13	36

	°	′	″
δ at G.M.N. =	21	3	7
Decrease in $13^m\ 36^s$ =			5·9
δ =	21	3	1·1
Approximately, z =	30	52	40
,, ϕ =	51	55	41

$$\text{The mean correction} = \frac{34{\cdot}90 \cos\phi \cos\delta}{\sin z} = 39{\cdot}1''$$

	m	s
E = sum of east hour angles =	17	36
W = sum of west hour angles =	10	14
E − W =	+7	22

$$d\delta = -\tfrac{26}{60} = -0{\cdot}43'' \text{ per minute}$$

∴ correction for change of declination (p. 188) $= -7\tfrac{1}{3} \times 0{\cdot}43 = -3{\cdot}2''$

	°	′	″
Mean observed altitude =	59	7	11·4
Mean correction =			39·1
	59	7	50·5
Refraction =			−33·5
	59	7	17·0
Parallax =			+4·5
	59	7	21·5
Declination (as above) =			−3·2
Mean meridian altitude =	59	7	18·3
z =	30	52	41·7
δ =	21	3	1·1
ϕ =	51	55	42·8 N.

Working by Least Squares.—In such cases there is always some uncertainty as to the exact time of transit, and the observations may be treated by the method of least squares.

Near the meridian, the apparent path of a heavenly body is practically parabolic, and may be represented by the equation

$$y = a + bt + ct^2$$

Any one observation is selected as origin (it is best to choose the one nearest to the mean of the observed times); the error in the observed altitude of that point is a; t is the time interval between this standard observation and any other, and is positive if the latter is the greater; y is the altitude difference between the standard observation and any other, plus if the latter is the greater; b and c are constants depending on latitude, declination, etc., and a, b, and c are to be found so as to give the parabola best fitting all the observations.

The formula for an error of observation is

$$e = a + bt + ct^2 - y$$

therefore the normal equations are (p. 47)

$$\begin{aligned} n \times a + b[t] + c[t^2] - [y] &= 0 \\ a[t] + b[t^2] + c[t^3] - [ty] &= 0 \\ a[t^2] + b[t^3] + c[t^4] - [t^2y] &= 0 \end{aligned}$$

where n is the number of observations, $[t]$ stands for the sum of all the values of t, and so on.

Working by this method, it is desirable to correct each altitude at the outset for change of declination.

The correction for each is *minus* $t \,.\, d\delta$, where $d\delta$ is the rate of *increase* of declination. This is to be added to all observations.

In this case, we take the fourth observation as origin or standard.

No.	t	$-t d\delta$	Observed altitude.	Corrected altitude.	
	m s	″			
1	− 6 52	− 3·0	59 6 1	59 5 58·0	− 1 49·0
2	− 4 57	− 1·7	59 6 28	59 6 26·3	− 1 20·7
3	− 3 3	− 1·0	59 7 30	59 7 29·0	− 18·0
4	0	0	59 7 47	59 7 47·0	0
5	+ 1 0	+ 0·3	59 7 37	59 7 37·3	− 9·7
6	+ 2 7	+ 0·7	59 7 49	59 7 49·7	+ 2·7
7	+ 4 14	+ 1·4	59 7 18	59 7 19·4	− 27·6
8	+ 5 37	+ 1·9	59 7 1	59 7 2·9	− 44·1

The values of t are found by subtracting the time of the fourth or standard observation (see table on p. 196) from each of the other times.

The value of $d\delta = -26''$ per hour (p. 196). Hence $-t \,.\, d\delta$ is found. The altitudes are taken from the table on p. 196, and $-t \,.\, d\delta$ is added in each case to give the corrected altitude.

Each value of y is found by subtracting the corrected altitude of No. 4 from each of the others. Reducing to seconds in all cases, we therefore have—

t	$t^2 \div 10$	$t^3 \div 10^3$	$t^4 \div 10^6$	y	yt	$yt^2 \div 10^3$
− 412	16,974	− 69,934	28,813	− 109·0	+ 44,908	− 18,502
− 297	8,821	− 26,198	7,781	− 80·7	+ 23,968	− 7,118
− 183	3,349	− 6,128	1,122	− 18·0	+ 3,294	− 603
0	0	0	0	0	0	0
+ 60	360	+ 216	13	− 9·7	− 582	− 35
+ 127	1,613	+ 2,048	260	+ 2·7	+ 343	+ 44
+ 254	6,452	+ 16,387	4,162	− 27·6	− 7,010	− 1,781
+ 337	11,357	+ 38,273	12,898	− 44·1	− 14,862	− 5,008
− 114	48,926	− 45,336	55,049	− 286·4	− 50,059	− 33,003

Hence the normal equations become—

$$8a - 114b + 489{,}260c + 286\cdot4 = 0$$
$$11\cdot4a - 48{,}926b + 4{,}533{,}600c + 5005\cdot9 = 0$$
$$48\cdot926a - 4533\cdot6b + 5{,}504{,}900c + 3300\cdot3 = 0$$

whence $a = -2\cdot22''$; $b = +0\cdot052054$; $c = -0\cdot00053679$

Now at the point of transit y should reach its maximum value, hence $\frac{dy}{dt} = 0$.

$$\text{But } \frac{dy}{dt} = b + 2ct$$

$$\therefore \text{ for point of transit } t = -\frac{b}{2c}$$

and if we substitute this in the equation $y = a + bt + ct^2$, we have

$$y = a - \frac{b^2}{4c}$$

With the above values of a, b, and c, $y = -0{\cdot}9''$.

This gives the correction to the standard altitude to find the meridian altitude.

The declination used in the final calculation is now that corresponding to the time of the standard observation which was taken as origin, viz. at $1^h\ 6^m\ 55^s$ watch time, or $0^h\ 12^m\ 55^s$ approximately G.M.T.

	°	′	″
Observed altitude of standard point =	59	7	47·0
y =			−0·9
Meridian altitude =	59	7	46·1
z =	30	52	13·9
Parallax and refraction =			29·0
Corrected z =	30	52	42·9
δ =	21	3	1·5
ϕ =	51	55	44·4 N.

The time of transit is $1^h\ 6^m\ 55^s - \frac{b}{2c} = 1^h\ 6^m\ 55^s + 48{\cdot}5 = 1^h\ 7^m\ 43{\cdot}5^s$, as against $1^h\ 7^m\ 36^s$ by plotting.

If the values of $a + bt + ct^2 - y$ be worked out for each value of t, the results will give the residual errors. These nearly balance, and the greatest residual is $+16''$, about, on the second observation.

This error is not excessive, though somewhat large, on a single reading. Each reading is subject to uncertainty from the semi-diameter correction (because different observers have different ideas as to when a hair appears to just touch the sun's disc); from level correction (one single division $= 20''$); from the circle reading (the smallest division $= 10''$); and from variations in the index error, in addition to errors of setting, etc. If all these happen to have the same sign for one particular reading, such an error as the above might easily result.

Determination of Azimuth

The next problem to be considered is that of finding the direction of the meridian, or true north-and-south line, through a given station. For this purpose we select some *other* station or fixed point on the survey, and observe the horizontal angle between it and some known star. At the same time we take such observations of the star as will enable us to calculate the horizontal angle between the star and the meridian.

These two horizontal angles combined clearly give the horizontal angle between the fixed survey line and the meridian.

The angle between the meridian of a station and a line through the station is called the *azimuth* of the line at that station. It may be measured from north through east right round to 360°, or from south through west, or on the four-quadrant system.

The fixed point of reference which marks the other end of the survey line is called the referring object.

Azimuth by Circumpolar at Elongation.—It will be evident from Fig. 77 that if the polar distance PS of a star be less than the co-latitude PZ, that star will culminate between the pole and the zenith. Its path is represented by a small circle like CSD. At C it is at upper culmination, and is then due north from Z, and therefore from the station (assuming that the elevated pole, P, is the *north* pole).

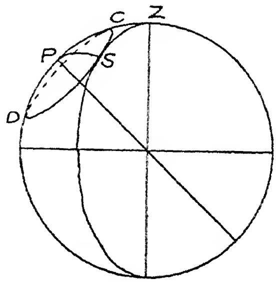


FIG. 77.

As it moves towards the west it will finally reach a point S in its path where a vertical circle, ZS, through S will be just tangent to the star's path. The star is then at its most westerly point, and is said to be at *western elongation*.

For a short time it appears to move quite vertically along ZS, and then moves back again towards the meridian, until it is once more due north at D.

It is clear that in the spherical triangle PZS the angle PSZ is 90°, because PS is at right angles to the small circle CSD, and therefore to ZS when the latter is tangent to the former.

The horizontal angle between the star and the meridian is the angle PZS of this triangle.

If the latitude, ϕ, of the station be first found by any of the methods already described, we have PZ $= 90° - \phi$; PS $= 90° -$ declination, and PSZ $= 90°$.

Hence, if the star be observed at the instant of elongation and the latitude be known, we have all the data for calculating the required angle PZS, which combined with the observed horizontal angle between the star and the referring object will give the azimuth of the latter.

This method has special advantages in azimuth determinations. The star is moving vertically; hence its horizontal angle with any mark can be observed more accurately than with a star which is moving rapidly horizontally. It is not necessary to observe the altitude at all, so that there is no chance of error from an uncertainty of the refraction correction.

It suffers from the disadvantages that we must wait for the elongation of a suitable star, and that for each star only one observation can in general be made, so that we cannot eliminate the instrumental errors by reversing face.

If, however, the local sidereal time be accurately known, it is possible to take two readings with opposite faces during a period of say two minutes, one just before and one just after elongation. During this period the star's motion is practically vertical (especially if the star is not more than a few degrees from the pole), so that the horizontal angle does not change.

Otherwise, for a good result two stars of nearly the same declination should be chosen (so that they shall have about the same altitude at elongation), and the face should be reversed after the first observation. Or the same star may be observed in this way both at eastern and western elongations, and the mean result taken for the azimuth of the referring object.

Suitable constellations for the latitude of England are Ursæ Minoris, Draco, Cepheus, Cassiopeia, and Ursæ Majoris. The nearer the star to the pole the better the result.

It is possible to calculate the time, altitude, and azimuth of the star beforehand; hence, if the direction of the meridian be approximately known, it is possible to set the telescope well enough to ensure that the selected star shall be visible in the field at the proper time.

Thus faint stars can be observed as before, and in this way many of the best results for azimuth have been obtained.

In the absence of an exact knowledge of time (local sidereal time is what is required), we must set the telescope on the star some minutes before elongation is expected. The star should be moving away from the elevated pole, and therefore should appear in the instrument to be moving towards it. As long as it does so, we follow it with the horizontal tangent screw until for some time it moves vertically and then begins to move back. It is then followed no longer, but the angle is read.

A note is made in the field book as to *which* elongation was observed.

Examples.—1. If the following observations of α Ursæ Majoris at elongation were made on March 15 and March 16, 1912, at a station whose latitude was 51° 21′ N., find the azimuth of the referring object (R.O.). $\delta = 62° 13' 34''$ N.

Body observed.	Face.	Reading of horizontal circle.	Remarks.
		° ′ ″	
R.O.	L.	276 18 25	Eastern elongation.
Star		315 43 40	
R.O.	R.	273 57 21	Western elongation.
Star		216 52 4	

In the right-angled triangle PZS (Fig. 77) we have $PS = 90° - \delta$; $PZ = 90° - \phi$; $S = 90°$.

It is required to find the azimuth angle PZS between the star and the meridian.

The formula is $\sin Z = \dfrac{\cos \delta}{\cos \phi}$.

The student should derive this formula for himself, working as in the examples on p. 36.

Substituting the above values of δ and ϕ—

$$Z = 48° \ 15' \ 12''$$

This is east or west of the meridian, according to which elongation is required.

For the eastern elongation—

	°	′	″
Horizontal angle to star =	315	43	40
„ „ R.O. =	276	18	25
Clockwise angle, R.O. to star =	39	25	15

The angle is clockwise. R.O. to star, as the reading to R.O. is the smaller.

It is as well now to draw a diagram (Fig. 78). Z is the station. Draw a line QZP to represent the meridian, ZP being the direction of the elevated pole, in this case the *north* pole.

Hence the *right* side of the figure represents *east*.

Set off $PZS = 48° \ 15' \ 12''$, the calculated angle at elevation, to the *east*, as we are considering eastern elongation.

Then ZS marks the direction of the star.

Now set off in the *anticlockwise* direction from ZS the angle $SZR = 39° \ 25' \ 15''$, being the observed clockwise angle R.O. to star.

ZR marks the direction of the referring object. And the diagram makes it clear that the angle between that direction and the meridian is PZS − RZS, and that the R.O. is east of the meridian.

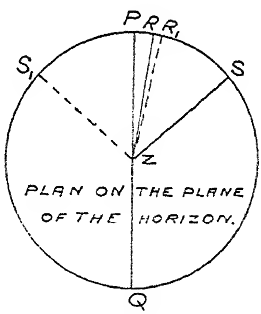


FIG. 78.

	°	′	″
PZS =	48	15	12
RZS =	39	25	15
Azimuth of R.O. =	8	49	57 *east* of north

For western elongation, PZS has the same value as before, but to the west.

	°	′	″
Horizontal angle to R.O. =	273	57	21
„ „ star =	216	52	4
Clockwise angle, star to R.O. =	57	5	17

The dotted lines on Fig. 78 refer to this case. R and R_1 should of course agree.

$$\begin{array}{rrrr} & {}^\circ & ' & '' \\ S_1ZR_1 = & 57 & 5 & 17 \\ S_1ZP = & 48 & 15 & 12 \\ \hline \therefore \text{ azimuth of R.O.} = & 8 & 50 & 5 \textit{ east} \\ \therefore \text{ mean azimuth of R.O.} = & 8 & 50 & 1 \ \text{,,} \end{array}$$

If it be desired to lay out the meridian on the field, this angle must be laid off *westwards* from the R.O.; that is, in the northern hemisphere from right to left. This gives the direction of due north.

More frequently the direction of the meridian is required *away* from the elevated pole. In this case set off the supplement of the calculated azimuth in the opposite direction.

(2) In the above example find the L.S.T. of each observation, and the altitude of the star.

First find the star's hour angle ZPS (Fig. 77).

Solving the triangle ZPS by the formulæ on p. 35, we have

$$\cos ZPS = \tan \phi \cot \delta$$
$$\therefore ZPS = 48^\circ\ 48'\ 39''$$

Reduce to time at 15° per hour.

$$\begin{array}{rrrr} & \text{h} & \text{m} & \text{s} \\ 48^\circ = & 3 & 12 & 0 \\ 48' = & & 3 & 12 \\ 39'' = & & & 2 \cdot 6 \\ \hline & 3 & 15 & 14 \cdot 6 \end{array}$$

This is east or west (that is, *minus* or *plus* respectively), according to the elongation considered.

$$\text{And L.S.T.} = \text{R.A.} + \text{hour angle}$$

$$\begin{array}{rrrr} & \text{h} & \text{m} & \text{s} \\ \text{From Nautical Almanac, R.A.} = & 10 & 58 & 22 \cdot 5 \\ \text{Eastern elongation, H.} = & -3 & 15 & 14 \cdot 6 \\ \hline \text{L.S.T.} = & 7 & 43 & 8 \text{ nearly} \\ \text{or, at western elongation,} & 14 & 13 & 37 \end{array}$$

The last figure was obtained by *adding* the hour angle to the right ascension.

On March 14th sidereal time is about one hour behind mean time.

Hence the L.M. times are about $8^h\ 45^m$ p.m. on March 15th, and about $15^h\ 15^m$, or $3^h\ 15^m$ a.m. on March 16th, civil date.

To find the altitude, ZS is required.

The formula is $\cos ZS = \dfrac{\sin \phi}{\sin \delta}$, which the student should verify

$$\therefore ZS = 28^\circ\ 2' \text{ nearly}$$
$$\text{or altitude} = 61^\circ\ 58'$$

This is only required when using a faint star, so that the theodolite cannot be directed to the star by eye. In this case the azimuth must also be calculated beforehand as above, and the approximate direction of the meridian must be known so that the angle can be laid off.

Azimuth by Extra-Meridian Altitudes.—The next best method for azimuth probably is by observing the altitude of a known star when well away from the meridian, the latitude having been previously found. At the same time, the horizontal angle between the star and some "referring object" is observed. The known altitude enables us to calculate the horizontal angle between star and meridian, and this, combined with the observed angle star to R.O., gives the angle between meridian and R.O.

This method has the great advantage that there need be no waiting. Any suitably placed star may be observed at the surveyor's chosen time, and the observations can be taken in pairs, reversing face between each pair so as to eliminate instrumental errors.

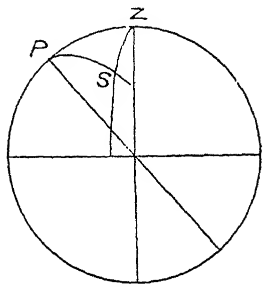


FIG. 79.

The mean readings of horizontal and vertical angles and time are taken for each pair, and the calculations worked from these. The time interval should not exceed about five minutes.

The method is subject to the disadvantage that the result is affected by any error in the refraction correction, but this is to some extent eliminated by observing two stars as nearly as possible symmetrically placed on opposite sides of the meridian.

In Fig. 79, let S be the star, then in the triangle ZPS

$$PZ = l = 90^\circ - \phi, \text{ and is known as } \phi \text{ is known}$$
$$PS = p = 90^\circ - \delta, \text{ and is known from N.A.}$$
$$ZS = z = 90^\circ - \text{altitude}$$

Thus the three sides are known, and we have the data for calculating the azimuth angle PZS between the star and the meridian.

This is east or west according to whether the star is rising or setting, and is combined with the observed horizontal angle by the aid of a diagram as before.

The observation may be made either on the stars or on the sun. If the latter, it is desirable if possible to make observations at about equal intervals of time before and after noon. The methods of observing described on p. 193 apply especially to this case.

Application to Time.—The same observation enables us to calculate the hour angle ZPS of the body observed, and hence the local time

As this is one of the most common methods of time deter-

mination, we shall in the example work both time and azimuth together.

If the time is not required, and the body be a star, the times of observation need not be observed. But if it be the sun the Greenwich mean time must be known, as already stated.

The formulæ of solution are given on p. 34, viz.—

$$\tan \tfrac{1}{2}(\text{ZPS}) = \sqrt{\frac{\sin(\text{S} - \text{PS}) \,.\, \sin(\text{S} - \text{PZ})}{\sin \text{S} \,.\, \sin(\text{S} - \text{ZS})}}$$

and

$$\tan \tfrac{1}{2}(\text{PZS}) = \sqrt{\frac{\sin(\text{S} - \text{ZS}) \,.\, \sin(\text{S} - \text{PZ})}{\sin \text{S} \,.\, \sin(\text{S} - \text{PS})}}$$

where S $= \frac{1}{2}$ sum of sides.

Example.—In latitude 32° 0′ 12″ on June 1, 1901, the following observations of α Coronæ were taken. The times were taken by a mean-time chronometer. Find the azimuth of the R.O., and if the approximate longitude in time be $4^{\text{h}}\ 30^{\text{m}}$ west, find the approximate error of the chronometer, on local mean time. The latitude was south.

Reading of vertical circle.	Level.		Face.	Time.	Horizontal angle.		Remarks.
	E.	O.			Star.	R.O.	
° ′ ″				h m s	° ′ ″	° ′ ″	
41 43 25	4	9	L.	10 22 14	78 43 4	213 51 6	Star rising
42 25 43	7	6	R.	10 26 24	260 14 40	33 51 10	

Barometer, 29·6 ins.; thermometer, 70° F.
$\delta = 38°\ 3'\ 18''$ S.; R.A. $= 19^{\text{h}}\ 2^{\text{m}}\ 48^{\text{s}}$; one division of bubble = 20″.

	° ′ ″
Mean vertical angle =	42 4 34
Level correction = $\dfrac{15 - 11}{4} \times 20'' =$	+20
	42 4 54
Refraction =	−1 0·5
Corrected altitude =	42 3 54 say
$z =$	47 56 6
$90° - \phi = l =$	57 59 48
$90° - \delta = p =$	51 56 42
2S =	157 52 36
S =	78 56 18

Note.—Here δ has been taken as positive, although *south*, because the latitude is also south.

° ′ ″	° ′ ″	° ′ ″
78 56 18	78 56 18	78 56 18
= 51 56 42	$z =$ 47 56 6	$l =$ 57 59 48
S − PS = 26 59 36	S − ZS = 31 0 12	S − PZ = 20 56 30

ANGLE PZS.

Angle.	Log sine.
S	9·9918558
S−PS	9·6569476
	19·6488034
colog	$\overline{20}$·3511966
S − ZS	9·7118814
S − PZ	9·5531755
2	$\overline{1}$·6162535
tan ½Z	$\overline{1}$·8081262

$$\therefore \tfrac{1}{2}Z = 32^\circ\ 44'\ 9''$$
$$Z = 65^\circ\ 28'\ 18''$$

ANGLE ZPS.

Angle.	Log sin.
S	9·9918558
S − ZS	9·7118814
	19·7037372
colog	$\overline{20}$·2962628
S − PS	9·6569476
S − PZ	9·5531755
2	$\overline{1}$·5063859
tan ½P	1·7531930

$$\tfrac{1}{2}P = 29^\circ\ 31'\ 52''$$
$$P = 59^\circ\ 3'\ 44''$$

To find the azimuth on the diagram (Fig. 80) let ZP be the direction of the elevated pole, which in this case is the *south* pole. As the star is *rising* it is to the *east*, and therefore on the *left* as we face south. Set off PZS as calculated above, accordingly.

To find the observed horizontal angle R.O. to star, some care is necessary. If we simply take the mean readings to R.O. and star and subtract, there is always a danger of an error of 180° in the result. It is perhaps wisest to obtain the angle separately with each face, and then take the mean.

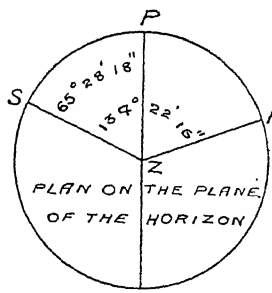


FIG. 80.

	°	′	″
Thus, face left, R.O. =	213	51	6
Star =	78	43	4
Clockwise angle, star to R.O. =	135	8	2
Face right, star =	260	14	40
R.O. =	33	51	10
Clockwise angle, R.O. to star =	226	23	30
,, ,, star to R.O. =	133	36	30
Mean angle, star to R.O. =	134	22	16

This being set off clockwise from ZS in the figure, it is clear that the referring object R is right—that is, *west*—of south.

	°	′	″	
SZR =	134	22	16	
SZP =	65	28	18	
PZR =	68	53	58	west of south

To find the time, reduce the hour angle P to time as before.

	h	m	s	
59° 3′ 44″ =	3	56	15	*east*, as the star is *rising*
∴ H =	20	3	45	west
R.A. =	19	2	48	
	39	6	33	
	24	0	0	
L.S.T. of observation =	15	6	33	
Longitude =	+4	20	0	
G.S.T. of observation =	19	26	33	
G.S.T. at G.M.N. =	4	37	5·7	from N.A.
Interval since G.M.N. =	14	49	27·3	

This is a sidereal interval. Reduced to mean time it gives—

	h	m	s
G.M.T. of observation =	14	47	2
Longitude =	−4	20	0
L.M.T. of observation =	10	27	2
Mean chronometer time =	10	24	19
Chronometer *slow* =		2	43

Here it is to be observed that the longitude has been applied first as a plus correction to reduce to G.S.T.; then the interval has been converted, and the longitude subtracted again. This part of the calculation could have been very slightly shortened, but the method given is easy to understand and as good as any.

Any error in the longitude affects the result to the extent of the *difference* between the corresponding solar and sidereal intervals.

Thus 1 sidereal minute = 59·8 mean solar seconds. Hence one minute of time error in the longitude affects the final result only by 0·2 second. Any small error in the latitude produces very little effect on the time determination if the star be very near the prime vertical. Its effect on the azimuth is, however, greater than that on time.

2. Take next the following observations of the sun on July 18, 1916, near Cheltenham, England.

Vertical angle.	Face.	Limb.	Watch time.	Horizontal angle. Sun.	Horizontal angle. R.O.
			h m s	° ′ ″	° ′ ″
42 22 16	L.	B.R.	4 23 0	181 27 30	
41 50 3	L.	T.L.	4 25 54	181 58 4	0 0 5
39 57 46	R.	B.R.	4 29 3	3 52 1	180 0 3
39 28 31	R.	T.L.	4 31 31	4 29 57	
40 54 39			4 27 22		

The angles are in all cases the mean of the two vernier readings, and the vertical angles have been corrected for level error. The watch was known to be $5^m\ 1^s$ slow on mean summer time on June 24, and to have a losing rate of about $1\cdot5^s$ per day.

The latitude was found from a circum-meridian observation on the same day as 51° 55′ 44″ N. The observation was an afternoon one. Barometer, 29·83″; thermometer, 68° F.

The student should carefully work through this example for himself.

	°	′	″
Mean observed altitude =	40	54	39
Mean refraction = 1′ 7″; thermometer, −2″ =		1	5
	40	53	34
Parallax (p. 195) =			6
Corrected altitude =	40	53	40
z =	49	6	20

Number of days from June 24 to July 18 = 24

∴ watch is slow on M.S.T. $5^m\ 1^s + 24 \times 1\cdot5 = 5^m\ 37^s$

∴ „ *fast* on G.M.T. . . . . $54^m\ 23$

	h	m	s
Mean watch time of observation =	4	27	22
		54	23
Approximate G.M.T. of observation =	3	32	59

	°	′	″
δ at G.M.N., from Nautical Almanac =	21	3	10
Decreasing 26·26 per hour; in $3^h\ 33^m$ =		−1	33
δ =	21	1	37 N.
p =	68	58	23
$90° - \phi = l$ =	38	4	16

(*Note.*—If the declination had been *south* (that is, opposite in name to the latitude), it would have been *added* to 90° to find p, the polar distance.)

The student should solve the triangle as before from the above values of the three sides z, p, and l. The results are—

	°	′	″
PZS =	109	37	40
ZPS =	49	42	37

In finding the azimuth PZS must be set off to the west (i.e. *left* of north), as the sun is *setting*.

To find the observed horizontal angle R.O. to star, note that we have clearly gone past 360° between the face left and face right readings to the sun (see table on p. 208). Hence we add 360° to each of the last two readings. We then obtain—

	°	′	″
Sum of readings to sun =	1091	47	32
Divide by four, mean =	272	56	53
Mean reading to R.O. =	90	0	4
Clockwise angle, R.O. to star =	182	56	49

On the diagram (which is left to the student) this is set off anticlockwise from the sun.

The result shows that the R.O. is *east* of north, 67° 25′ 31″.

To find the time, the hour angle ZPS is reduced to time as usual, and gives the *local apparent time*. This is corrected by the equation of time to give L.M.T.

		h m s
	∴ L.A.T. =	3 18 50·5
Equation of time = $6^m\ 0{\cdot}1^s$ at G.M.N., increasing $0{\cdot}19^s$ per hour, and to be added to apparent time. Therefore at $3^h\ 33^m$ G.M.T.	=	+6 0·8
	L.M.T. . .	3 24 51

As the watch gives *Greenwich* time the observations do not enable us to find its error, the longitude being unknown. But assuming that the Greenwich time given by watch, as above, is correct, we can find the longitude.

	h m s
G.M.T. by watch =	3 32 59
L.M.T. by observation =	3 24 51
Longitude in time =	8 8 *west*

Prime Vertical.—A vertical plane through the station at right angles to the meridian is called the *prime vertical* of the station.

Best Position for Observing.—The best position for the body observed in the above method of finding azimuth and time is, theoretically, on the prime vertical.

But it must be remembered that the lower the altitude of the body the more uncertain becomes the refraction correction.

It is desirable that the altitude should not be less than 35 or 40 degrees. To secure this it may be desirable, as in the above exercise, to observe 20°, or even 30°, away from the prime vertical.

Approximate Methods for Azimuth.—1. A star is on the meridian of a station, at upper culmination, when the L.S.T. is equal to the R.A., and at lower culmination when the difference between L.S.T. and R.A. is 12 hours.

Hence if the local sidereal time be known, and a star be observed at either of these times, its horizontal direction will coincide with that of the meridian, and will have the same name as the latitude (north or south) if the star is at lower culmination or between the zenith and the elevated pole; but the *opposite* name if the star culminates on the side of the zenith *away* from the elevated pole.

This method is best used with a star near the pole, like the pole star, as an error in the L.S.T. makes less error in the result.

2. If two stars have either the *same* right ascension, or if their R.A.'s differ by exactly 12 hours, then when these two stars are in one vertical plane, that plane will be the meridian plane.

If one star be near the pole the result is thereby improved.

Many years ago this condition was very nearly satisfied by the stars Polaris (the pole star) and ε Ursæ Majoris (the *third* star in the tail of the Great Bear), and this method of finding the meridian by the use of those two stars was given by Rankine in his "Civil Engineering," and ascribed to Mr. Butler Williams. Since then—largely owing to the rapid motion in right ascension of stars near the pole, already referred to (pp. 171 and 172)—these two stars have moved away from the same hour circle.

When the first edition of this book was being prepared the necessary condition was nearly satisfied by the pole star and the *second* star in the tail of the Great Bear.

The diagram given in Rankine was re-drawn accordingly, as shown in Fig. 81.

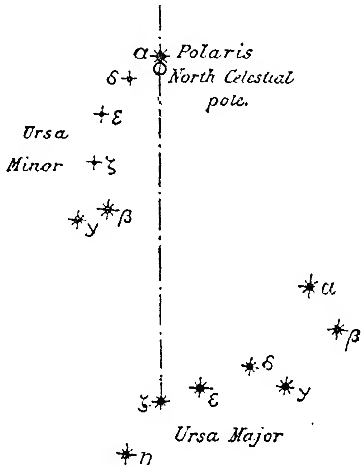


Fig. 81.

At the present time (1916) the hour circle through the pole star passes almost midway between the *first* and *second* stars in the tail of the Bear, the right ascensions being about

	h	m
Polaris	1	30
ζ Urs. Maj.	13	21
η „ „	13	44

The instant at which the stars are on the same vertical may be observed by the aid of a plumb-bob, or by other means which can be devised. It will, of course, nearly correspond with the instant at which the local sidereal time is equal to the R.A. of the stars, or differs therefrom by 12 hours.

The student can easily find suitable pairs of stars for use with this method by the aid of star charts. For example, Castor and Procyon. These differ in R.A. by about five minutes. When they are on one vertical, that vertical plane is about 4° 15′ from the meridian in latitude 51° N., which would be good enough for giving the direction by eye.

(3) *By equal altitudes of any star.*

If the two readings of the horizontal circle be taken when the telescope is set on the same star when rising and when setting, the altitude being the same at both settings, the mean of the readings will give the direction of the meridian. The method is not very exact with the sun in consequence of the change of declination.

In any case it is only approximate, because if the vertical circle remains clamped, as usually recommended, there are no facilities for reversing face, and if it be unclamped it is very difficult to set exactly the same angle again.

As a means of finding local time, the method is not subject to most of these disadvantages.

Time by Equal Altitudes of a Star.—If a star (the declination of which is constant during the two observations) be observed before and after transit over the meridian when at the same altitude, it is obvious that its hour angle at both instants of observation must be equal to each other though on opposite sides of the meridian, and that the mean of these times of observation will give the actual time that the star was on the meridian.

As a star rises in altitude we may record an instant (say t) which by our timekeeper corresponds to the time that the star reaches a certain altitude, then, by waiting till the star has crossed the meridian and descends to the same altitude, and again recording the instant (say t'), we can find the moment of transit, which was $\frac{t + t'}{2}$. A comparison of this time (reduced to S.T.) with the R.A. of the star, determines for us the error of our timekeeper.

This method of equal altitudes is very accurate, and is independent of an accurate knowledge of latitude or declination, or of errors in collimation, or in the graduation of the instrument. It is (with a star) very simple in its procedure, but requires a clear sky for a considerable interval of time, and for altitudes less than about 45° it is liable to variations on account of changes in temperature, and therefore of "refraction."

Time by Equal Altitudes of the Sun.—When the sun is used, a correction has to be made for the change of declination that occurs between the times of observation before and after the passage over the meridian.

Having obtained the "mean of the two times" of observation, it may be corrected to the "time of apparent noon," *i.e.* time of transit, by the following formula:—

$$\text{Mean of times} \pm \frac{t \times d\delta}{15}(\tan \phi \operatorname{cosec} t - \tan \delta \cot t)$$

where t = half the interval between the two observations,
$d\delta$ = the change of declination in one hour,
δ = the declination of the sun when on the meridian.

This correction is *positive* when the sun is *leaving* the "elevated pole," in declination, *negative* when it is *approaching* the "elevated pole."

Instructions for Observing.—When observing for equal altitudes, the theodolite is clamped at the proper angle for the second observation.

The star must be watched when the proper time approaches, until it enters the field, after which it must be kept on the vertical hair—by working the horizontal tangent screw—until it reaches the intersection of the hairs.

If the sun be the body observed, the *upper* limb must be observed (unless artificial illumination is provided for the cross hairs), because with the ordinary dark glass the hairs are not visible until actually covered by the sun's image, so that in the afternoon we should not see the hair until after the lower limb had crossed it.

Time by Meridian Transits.—If the direction of the meridian be very accurately known, there can be no doubt that the best method for time is by taking transits over this meridian. We have seen that the local sidereal time at which any star transits is equal to the star's right ascension.

When the meridian is approximately known, a series of observations on different stars enables us to find the "deviation error" (*i.e.* the angle between the true meridian and our assumed one) with great accuracy, and we can then allow for this.

This is the method for time most frequently used for very exact determinations, but is too tedious for ordinary surveying work, and the student is referred to works on Astronomy for a full account of it.

Longitude Determination.—The determination of the longitude of a station amounts to finding both the Greenwich and the local time at one and the same instant. The difference gives the longitude expressed in time, and this is reduced to arc at 15° per hour. Similarly, the difference between the local times at any two stations at one and the same instant gives the difference of longitude between the stations. The time used may be mean solar or sidereal, but both times must be in the same units.

By Electric Signals.—The most accurate method of determining time, as already stated, is by a number of meridian transits, the exact direction of the meridian having been very carefully fixed.

Hence the most exact method for difference of longitude is as follows: Let A and B be the two stations; the observer at A

takes the time of transit, over his meridian, of a certain star by his clock, and at the same instant sends a signal to the observer at B, who also notes the time by his clock. These signals may be sent either by wireless or through wires specially laid.

When the same star transits at B, the observer there notes the time by his clock and sends a signal to A, who also notes the time. The *rate* of each clock is found either by observations of the same star on successive evenings, or of similarly placed stars on the same evening. The time interval by each clock, after correction for rate, should be very nearly the same, and the mean—reduced to a sidereal interval if read on mean-time clocks—gives a measure of the longitude. The observations should be repeated on stars in different positions, unless the deviation and other errors of each instrument are accurately known beforehand.

If the sun is the body observed, the time interval must be corrected for the change in the equation of time between the transits.

The disadvantages of this method are that it requires at least two skilled observers, and a considerable outlay for apparatus for sending the signals. It is beyond the reach of the ordinary surveyor.

The local time is found, therefore, by any of the methods described above (an extra meridian altitude is usually most convenient), and the real difficulty in longitude determination is to find the Greenwich time. Some methods of doing this follow.

Transportation of Chronometers.—The surveyor wishing to determine longitude at all accurately would generally carry at least two chronometers whose errors on G.M.T. would be found at the nearest station of known longitude, and carefully rated under conditions as nearly as possible the same as those of the survey.

The rate can be constantly checked wherever the observer remains at the same station for some hours. In this way Greenwich time is always fairly well known.

In the years 1849–55, fifty chronometers were transported in finding the difference of longitude between Liverpool and Cambridge in the United States. The *mean* obtained by the voyages in each direction differed by only $0{\cdot}15^{s}$ of time.

Moon-culminating Stars.—If the chronometers appear to develop a variable rate, or for any reason are not considered reliable, the position of the moon gives a measure of the Greenwich time. Assuming that it has been possible to lay out the meridian pretty accurately, perhaps the best method is to take the transit of the moon and of a star which is quite close to it.

The Nautical Almanac, under the head of "Moon-culminating Stars," gives suitable stars for every day in the year, and the figures

necessary for finding the longitude, with worked examples in the "explanation."

Unfortunately the stars are usually too faint to be visible by eye, and the telescope must be *set* for observing them. For this the latitude must be known.

Then by the formula on p. 179, we have

$$z = \phi - \delta$$
$$\text{and altitude} = 90° - z$$

If the telescope is set to this altitude in the meridian, the star should appear in the field shortly before the L.S.T. becomes equal to its right ascension.

Example.—On May 26, 1912, the times of transit of the moon's bright limb and the star α Virginis were observed as in the table below.

Find the longitude, given that the times were by mean-time watch, the rate being negligible for the interval considered. Also find the G.M.T.

	Time of transit.
	h m s
Moon	8 23 31·2
Star	8 44 35·8

Here it is to be remembered that the local sidereal time at the instant when any body crosses the meridian is equal to the R.A. of the body. Hence the *difference* between the times of transit of two bodies tells their *difference* of R.A., assuming that both right ascensions remain fixed. In this case the *star's* right ascension may be regarded as constant, but the moon's varies, hence the difference of the observed times tells the difference of R.A. *at the instant of the moon's transit*, and therefore the R.A. of the moon's bright limb at that instant, as that of the star is known. But the Nautical Almanac gives the R.A. of the bright limb at transit at Greenwich (see under "Moon-culminating Stars") and the change in R.A. per hour of longitude. Hence we have the data for the longitude:

	h	m	s	
Diff. of observed times =		21	4·6	mean time
By conversion tables =		21	7·1	sidereal
R.A. of star from N.A. =	13	20	35·0	
By subtr. R.A. of moon =	12	59	27·9	
But R.A. at Gr. Transit =	13	0	27·7	
By subtr. change of R.A. =			59·8	
But change per hour =			121·6	

$$\therefore \text{ Longitude in time} = \frac{59{\cdot}8}{121{\cdot}6} \times 1 \text{ hour}$$
$$= 29^{\mathrm{m}}\ 30{\cdot}4^{\mathrm{s}}\ \textit{east}$$

We *subtract* the time interval from the star's R.A. because the moon transits first, and the longitude is *east* because the moon's R.A. is *less* at the place than at Greenwich.

If the difference of longitude is great, the *average* rate of increase in the moon's R.A. during the period between the Greenwich and local transits should be calculated from the rates at two successive Greenwich transits, and used for the final result. The greatest source of error is the difficulty of saying exactly when the hair is tangent to the moon's disk.

One second of error in the observed time interval causes an error of about 30 secs. of time in the longitude.

By Lunar Distances.—Let MS (Fig. 82) represent the moon and a distant star. In the triangle PSM, PS and PM are the complements of the declinations, and MPS is the difference of the right ascensions.

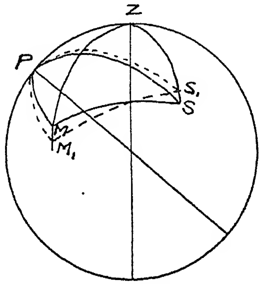


FIG. 82.

Hence, if the declinations and R.A.'s are known, we can solve the triangle to find the side MS, which is the angular distance between the moon and the star, as measured at the centre of the sphere.

Now the Nautical Almanac gives the declinations and right ascensions when the Greenwich time is known. Hence for each Greenwich time there is a corresponding value of the distance MS, which can be calculated

Further, as the moon changes rapidly in position while the star is nearly fixed, the distance varies fairly rapidly, and it is clear that if we could actually measure the distance MS with a sextant at any instant, it would be a simple matter to find the corresponding Greenwich time.

The difficulty arises in the fact that the data in the Nautical Almanac refer to the moon's position as seen from the earth's *centre*, whereas we observe from the surface. There is therefore a parallax correction introduced which is considerable in consequence of the moon's nearness. Moreover, the apparent distance is affected by refraction.

To correct accurately for the moon's parallax is a complicated business, as it is necessary to take the spheroidal shape of the earth into account. But such refinement is scarcely necessary with ordinary sextant observations. The value of the moon's horizontal parallax is given in the Nautical Almanac (p. iii. of each month), and we shall assume that the parallax always simply makes the apparent altitude too small by an amount $m \cos h$, where m is the horizontal parallax, and h the altitude (see p. 195).

We shall also assume latitude and local time known, as well as the approximate longitude.

Let M_1 be the apparent position of the moon, the displacement MM_1 being due to parallax and refraction, of which the former is the greater. And let S_1 be the displaced position of the star due to refraction.

The apparent distance is M_1S_1, and this is supposed to be accurately measured with a sextant, the time being also noted.

Then from the known latitude and time we first calculate the true zenith distances ZM and ZS. Apply the corrections for parallax and refraction to find ZS_1 and ZM_1; from these and the measured distance M_1S_1 calculate the angle M_1ZS_1; then from this and the known value of ZM, ZS calculate the true distance MS.

This process is called "clearing the distance." From the cleared distance the Greenwich time is found by interpolation.

This method of finding Greenwich time has the advantage over moon-culminating stars of requiring much less "preparation." It is not necessary to lay out the direction of the meridian, or to wait for the moon's transit, which may occur at an inconvenient time. The lunar distance may be measured at any time. It gives a less reliable result, however.

In observing, a dark glass should be placed over the mirror through which the moon is seen, to reduce the brightness and make the star more distinctly visible, and some little practice is necessary for bringing the star over to the moon (p. 13) so as to be sure that the right star is observed.

In the plane containing the moon and star there is always *one* edge or limit of the moon which is *complete* (that is, which represents the true edge of the moon, and not merely the edge of the part illuminated by the sun). The distance is, of course, observed to this edge, the tangent screw being adjusted until the star is exactly on the edge.

Hence the observed distance must be corrected for semi-diameter (given in Nautical Almanac), the correction being added or subtracted according to whether the complete edge was the *nearer* edge to the star or not.

The value of the apparent semi-diameter is, with sufficient accuracy, found by the formula $D \times \frac{\sin z_1}{\sin z}$, where D is the moon's semi-diameter for the centre of the earth as given in the Nautical Almanac, z is the true zenith distance ZM and z_1 is the zenith distance as affected by *parallax*, but *not* by refraction.

If the apparent altitudes can be observed at the same time as

the lunar distance, the calculation is thereby shortened, as M_1Z and S_1Z are known at once, and MZ and SZ are found from them by correcting for parallax and refraction.

Any error in the assumed Greenwich time, then, moreover affects the results to a much smaller extent, so that we get a closer approximation to the truth. If this is to be done by one observer, however, he must be very skilful, as he must read two altitudes of the moon and star just before reading the distance, and two just after, and, say, at least two readings of the distance. All these must be done within 10 or 15 minutes for interpolation.

Example.—On November 3, 1916, the distance between Altair and the nearer limb of the moon was observed with a sextant as 35° 25′ 5″; the time by L.M.T. clock was $6^h\ 45^m\ 13^s$, and the approximate longitude—in time—was $29^m\ 48^s$ W. The time has been corrected for clock error, and the distance for index error. Find the G.M.T. and the true longitude. The latitude of the station was 54° 20′ 10″ N.

The first step is to calculate the true altitudes.

In the triangle ZPM

$$\cos ZM = \cos PZ \cos PM + \sin PZ \sin PM \cos ZPM$$

Here ZM is the true zenith distance, PZ the co-latitude, PM the moon's polar distance, and ZPM her hour angle.

	h	m	s
Now L.M.T. =	6	45	13
Approximate longitude		+29	48
Approximate G.M.T. =	7	15	1
G.S.T. at G.M.N. =	14	49	38·7
whence G.S.T. of observation =	22	5	51·2
and L.S.T. =	21	36	3

	°	′	″
Now moon's δ at 7 p.m. =	10	46	4 S.
Decreasing $140{\cdot}4^s$ in 10^m ∴ in $15^m\ 1^s$ =		3	31
∴ δ at $7^h\ 15^m\ 1^s$ =	10	42	33 S.
∴ PM =	100	42	33

as P is the *north* pole.

	h	m	s
Moon's R.A. at 7 p.m. =	21	44	22·5
Increasing $21{\cdot}8^s$ in 10^m ∴ in $15^m\ 1^s$ =			32·7
∴ R.A. at $7^h\ 15^m\ 1^s$ =	21	44	55
L.S.T. =	21	36	3
∴ ZPM in time =		8	52 *east*

or ZPM in arc = 2° 13′

and λ = 54° 20′ 10″ N.
∴ PZ = 35 39 50

Substituting in the equation for cos ZM, we find ZM = 65° 4′ 21″.

Similarly ZS is found from the triangle ZPS, except that we find at once from the Nautical Almanac for the star Altair,

$$\begin{aligned} & \qquad {}^{\circ}\quad{}'\quad{}'' \\ \delta &= \ \ 8\ 39\ 10\text{ N.} \\ \therefore\ \text{PS} &= 81\ 20\ 50 \\ & \qquad \text{h}\quad \text{m}\quad \text{s} \\ \text{and R.A.} &= 19\ 46\ 44 \\ \text{whence ZPS} &= \ \ 1\ 49\ 19\ \textit{west} \\ & \qquad {}^{\circ}\quad{}'\quad{}'' \\ \text{or, in arc,} &= 27\ 19\ 45 \\ \text{These give ZS} &= 50\ 38\ \ 2 \end{aligned}$$

These figures have not been checked, and the example should be carefully worked by the student.

All the working down to this point would be saved by observing the altitudes.

The next step is to correct the true zenith distances for parallax and refraction, so as to find the apparent values. Parallax is to be *added* to the zenith distance, and refraction *subtracted.*

If the altitudes had been observed this step would have been reversed, to find the *true* zenith distance from the observed, and the signs of the corrections would then be reversed too, of course (pp. 195 and 178).

From Nautical Almanac, moon's horizontal parallax = 58′ 58″. Multiply by cos h or sin ZM.

	°	′	″
Therefore parallax =		53	28
add ZM =	65	4	21
	65	57	49
Subtract refraction		2	9
ZM_1 =	65	55	40

	°	′	″
For the star, ZS =	50	38	2
Subtract refraction		1	10
ZS_1 =	50	36	52

The next step is to correct the observed distance for the moon's semi-diameter.

Semi-diameter, from Nautical Almanac, = 16′ 6″

Multiply by $\dfrac{\sin ZM_1}{\sin ZM}$ (p. 217), where ZM_1 is the true zenith distance affected by parallax, but not by refraction; in this case, 65° 57′ 49″.

	°	′	″
Therefore apparent semi-diameter =		16	13
Add observed distance =	35	25	5
$\therefore S_1M_1$ =	35	41	18

Here we *add*, because the distance was observed to the *nearer* limb of the moon.

We now know the three sides of the ZM_1S_1, and we find the angle Z.

$$\tan \tfrac{1}{2}Z = \sqrt{\frac{\sin (s - ZM_1) \sin (s - ZS_1)}{\sin s \, . \sin (s - S_1M_1)}}$$

This gives $Z = 38° \, 21' \, 16''$

Now with this value of Z and the known values of ZM and ZS, calculate the true distance SM.

$$\cos SM = \cos ZM \cos ZS + \sin ZM \sin ZS \cos Z$$

This gives $SM = 35° \, 12' \, 12''$

This is the "cleared distance."

Finally, we must determine the Greenwich time corresponding to this.

Take an approximate G.M.T., say $7^h \, 15^m$, and from Nautical Almanac take out the declinations and right ascensions of the moon and star.

The declinations give PS and PM, and the difference of right ascensions gives the angle SPM. Then solve the triangle PSM to find SM.

	°	′	″
As above, at $7^h \, 15^m$ G.M.T. PM =	100	42	33
and PS =	81	20	50

	h	m	s
Moon's R.A. =	21	44	55
Star's R.A. =	19	46	44
SPM =	1	58	11

or in arc $= 29° \, 32' \, 45''$

$$\cos SM = \cos PS \cos PM + \sin PS \sin PM \, . \cos SPM$$

This gives $SM = 35° \, 12' \, 3''$

This is too *small* by 9″, as compared with the cleared distance. Hence we try a *later* G.M.T. (as the moon is behind the star) say 1 minute later, as the result is so near. Find the corrected R.A. and δ for the moon, and hence new values of PM and SPM, and recompute SM. This gives

	°	′	″
PM =	100	42	19·5
SPM =	29	33	21
and therefore SM =	35	12	25

Hence for 1 minute of time SM has increased 22″. In how long will it increase 9″? The answer is, in 25^s nearly.

	h	m	s
Therefore corrected G.M.T. =	7	15	25
L.M.T. =	6	45	13
Longitude in time =		30	12 W.

$= 7° \, 33'$ W.

If the computed G.M.T. were very different from that originally taken as an approximate value, the whole calculation should be repeated with the new value.

Examples for Exercise.

(1) Determine the latitude of the place in which the following observations were made with a zenith instrument by Talcott's method. The values

of δ for the two stars were $+52° 48' 47 \cdot 43''$ and $+32° 40' 9 \cdot 03''$, respectively. The micrometer used read towards the zenith, and one revolution of its screw $= 43 \cdot 64''$. One division of the level scale $= 1''$, the graduations being numbered from the centre outwards. The micrometer reading for the south star was $19 \cdot 065$, and the level reading $33 \cdot 0$ N. and $34 \cdot 8$ S.; for the north star the micrometer reading was $17 \cdot 365$, and the level readings $34 \cdot 5$ N. and $33 \cdot 2$ S. The effect of refraction may be neglected. (Survey Certificate Examination, Cape of Good Hope, 1904.)

Ans. $42° 43' 51 \cdot 01''$ N.

(2) The readings of the horizontal circle of a theodolite when pointed (*a*) on a distant fixed mark, (*b*) on a star whose south polar distance is $10° 40' 23''$ at the instant when it is at its greatest elongation, were respectively $79° 24' 30''$ and $263° 21' 20''$. The readings increase as the telescope is turned from N. to S. through E. Determine the true azimuth of the distant mark, the observer's latitude being $32° 47' 50''$ S. (Same examination, 1905.)

Ans. $163° 19' 29''$ W. of S. if at eastern elongation, $188° 46' 51''$ W. of S. if at western.

(3) Explain what is meant by the equation of time. Determine the mean time at which *m* Virginis crosses the meridian on May 31st, if the R.A. $= 13^h 36^m 33 \cdot 69^s$, and the sidereal time at the preceding mean noon $= 4^h 27^m 17 \cdot 46^s$. (Same examination, 1904.)

CHAPTER VI

THE FIGURE OF THE EARTH

The Earth's Geodetic Surface.—When we speak of the "earth" as a sphere or spheroid, we do not mean that the external or visible surface of the earth is such, but that the *mean* "surface of the sea" (produced in imagination so as to percolate the continents) forms a regular surface of revolution. We may imagine the continents traversed in every direction by subterranean channels, below sea-level, to which the sea has complete access, and further, that numerous wells or shafts are sunk and connected to this underground network of channels. Then the mean level of the water in the wells would coincide with the "geodetic surface" of the "earth." This "surface" is that to which all "geodetic measurements" are referred.

"Levels," or "altitudes" of terrestrial places, are the vertical distances between the "geodetic surface" and the actual surface of the ground. "Geodetic distances" are measured on the "geodetic surface."

In consequence of the perturbing influences of large land masses and other causes, it is, of course, highly improbable that the mean actual surface of the sea, found as described above, would form an exact surface of revolution. The most we can hope for is to find a surface which shall not seriously depart from the true geodetic surface as above defined.

General Form of Geodetic Surface.—The general form of the earth, as bounded by the "geodetic surface," is an oblate spheroid, that is to say, the solid generated by the revolution of an ellipse about its shorter or minor axis. The difference in length of the two axes of this ellipse is small, the ratio being approximately as 300 is to 301. For many purposes, therefore, the "geodetic surface" may be treated as a sphere.

Effect on the Direction of the Force of Gravity.—One important property of the "geodetic surface" is that a "plumb-line," suspended above any point in it, will be normal to that "surface," that is to say, the plumb-line will be perpendicular to a plane

tangential to the surface at the point immediately below the "plumb-line."

Now, the "plumb-line" indicates the direction of the resultant of the force of gravity, that is to say, the direction of the resultant of the attractions of the particles of matter which form the mass of the earth, combined with the centrifugal force due to the earth's rotation, and it is essential to the stability of the figure of the earth that this should be normal to the "geodetic surface."

The Polar Axis.—The shorter axis of the ellipse, which by its revolution generates the ellipsoid that coincides with the geodetic "surface" of the "earth" is the "polar axis." This is the axis about which the earth rotates daily. Hence any section containing the polar axis is an ellipse.

Geocentric Latitude.—The angle which a line from the "centre of the earth" to any point in its surface makes with the "plane of the equator" is called the "geocentric latitude" of that point. Were the earth a true sphere, "geocentric latitude" would serve as a definition of "latitude" generally, but such is not the case.

Geographical Latitude defined.—The "geographical latitude" of a point on the "earth's surface" may be defined as the "angle"

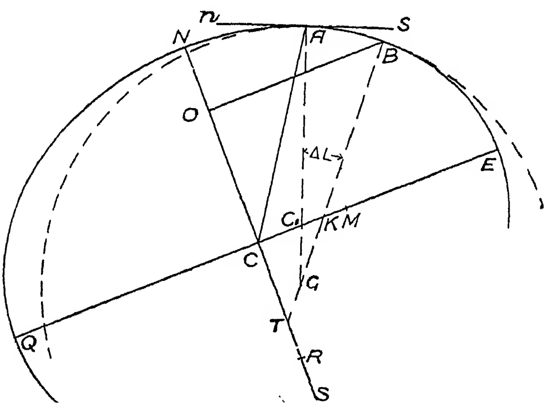


FIG. 83.

which a "plumb-line," suspended over the point in question, produced makes with the "plane of the equator."

Thus, referring to Fig. 83, let C be the centre of the earth, NCS its axis, and EQ the equator whose plane is at right angles to the axis.

Let A be the point on the earth's surface whose latitude is required.

If the earth were a true sphere, then CA would be the direction of the plumb-line at A, and if prolonged would pass through C, the centre. If it be not a true sphere, but an ellipsoid, then the line GA, representing the direction of the plumb-line, would pass through some other point C_1 in the plane of the equator. In either case the angle ACE or AC_1E, as the case may be, is the latitude of the point A. The angle ACE is termed the "geocentric latitude," whilst AC_1E is termed the "geographical latitude."

As the second case represents the true state of affairs, the latter angle is that which is found in the astronomical determination of latitude as described in Chapter V., and is what is usually understood as latitude. The direction of the zenith at A is given by GA, and the spirit level of the theodolite should indicate a direction at right angles to this.

Again, let nAs be the projection of a plane touching the earth at A. This plane is called the plane of the sensible horizon, and the plumb-line is perpendicular to it. The line of collimation of a correctly adjusted level set up at A is exactly parallel to the plane of the sensible horizon in every direction.

We shall now consider the evidence afforded by latitude determinations as to the true figure of the earth.

Radius of Curvature.—If a, b be the semi-axes of an ellipse parallel to the axes of x and y respectively, the equation to the curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hence

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

and

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$$

Therefore if ρ be the radius of curvature at any point, we have

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}$$

Substitute for y^2 from the general equation, and we have

$$\rho = \frac{\{a^4 - x^2(a^2 - b^2)\}^{\frac{3}{2}}}{a^4b}$$

If we assume that the section of the earth along any meridian is an ellipse, the equatorial and polar semi-axes being a and b respectively, this formula therefore gives the radius of curvature along the meridian at any point; that is, the radius of the circle which most nearly coincides with the ellipse at the point considered.

If we put $b = a$ we obtain the result on the assumption that the earth is a sphere, and we have, in this case, $\rho = a$ for all values of x.

If we put $x = a$ in the equation for ρ, we have

$$\rho = \frac{a^3b^3}{a^4b} = \frac{b^2}{a}$$

This gives the radius of curvature along the meridian at the equator. It is shown by EM in Fig. 83.

If $x = 0, \rho = \frac{a^2}{b}$, giving the radius of curvature at the pole. This is shown by NR in Fig. 83.

Now if we assume the polar axis less than the equatorial (i.e. b less than a), which is the condition that the earth may be an oblate spheroid, then $\frac{b^2}{a}$ is less than $\frac{a^2}{b}$, or the radius of curvature is less at the equator than at the pole.

Hence we can test the assumption by finding the radii of curvature at different points.

Now suppose that observations for latitude were made at points A and B (Fig. 83), not very far (but due north and south) from each other; thus the difference of latitude is known. Let this be ΔL, and let D be the geodetic distance between the points found from a triangulation survey.

Then if G be the centre of curvature at the middle point of AB, GA or GB will give the radius of curvature very nearly for the short length AB. That is, AB may be regarded as an arc of the circle (shown by a broken line in Fig. 83) with centre at G.

Now AC_1E and BKE are the latitudes of A and B respectively, and it is clear that the angle AGB is the difference between these, or ΔL. Hence if r = AG or BG, we have

$$\mathrm{AB} = \mathrm{D} = \frac{2\pi r \times \Delta \mathrm{L}}{360^\circ}$$

where ΔL is in degrees;

$$\therefore r = \frac{\mathrm{D} \times 360}{2\pi \times \Delta \mathrm{L}}$$

whence r is known for the middle latitude of A and B.

Such determinations show that as we approach the poles the radius of curvature increases, decreasing towards the equator, and therefore verifying the assumption of an oblate spheroid.

Taking again the equation for ρ, viz.—

$$\rho = \frac{\{a^4 - x^2(a^2 - b^2)\}^{\frac{3}{2}}}{a^4 b}$$

it is shown later (p. 227) that

$$x^2 = \frac{a^4}{b^2 \tan^2 \lambda + a^2}$$

where λ is the latitude.

Substituting, we have

$$\rho = \frac{a^2 b^2 \sec^3 \lambda}{(b^2 \tan^2 \lambda + a^2)^{\frac{3}{2}}}$$

Thus any two values of ρ determined as above in known latitudes would enable us to find values for a and b. Actually, of course, many determinations must be made, and treated by least squares or otherwise so as to find the best values.

Longitude.—Again, in Fig. 84 let the full ellipse represent the section along the meridian through a point A on the earth's

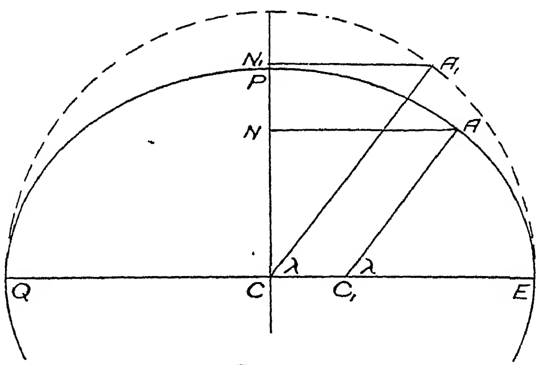


Fig. 84.

surface, AC_1 the normal or vertical at A, and AN a plane perpendicular to the axis CN.

Then AC_1E or λ is the latitude of A, and AN is the radius of the parallel of latitude through A. Hence, in the latitude of A the length corresponding to 1° of longitude will be the 360th part of the circumference of a circle of radius NA.

Now let the broken circle represent the section of the earth assumed spherical and of the same radius as the equator CE. From C draw CA_1 parallel to C_1A; then A_1 is the point whose latitude is λ on this assumption. N_1A_1 is the radius of the parallel for the latitude λ, and the length of 1 degree of longitude will be the 360th part of this circumference.

Actually we can measure the length of a degree of longitude in different latitudes, and so arrive at a conclusion as to which assumption is correct.

To find expressions for the radii AN and A_1N_1, let CE be the axis of x, $CE = a$, $CP = b$. Then the equation to the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Again, as C_1A is perpendicular to the tangent,

$\tan \lambda = -\frac{dx}{dy} = \frac{a^2}{b^2} \cdot \frac{y}{x}$ from the above equation. Hence

$y = \frac{b^2 x \tan \lambda}{a^2}$, and substituting in the general equation we obtain

$$\frac{x^2}{a^4}(b^2 \tan^2 \lambda + a^2) = 1$$

whence $$x^2 = \frac{a^4}{b^2 \tan^2 \lambda + a^2}, \text{ where } x = AN$$

If we put $b = a$ in this, we obtain

$$x_1{}^2 = \frac{a^4}{a^2(1 + \tan^2 \lambda)} = \frac{a^2}{\sec^2 \lambda} = a^2 \cos^2 \lambda, \text{ where } x_1 = A_1N_1$$

Now if the earth is an oblate spheroid, b or CP is less than a or CE. Hence, by putting $b = a$ in the equation for x^2, we *increase the denominator* and therefore *decrease* the value of x^2. That is, if the earth is an oblate spheroid x^2 is *greater than* $x_1{}^2$, or the length of a degree of a longitude in any given latitude must be *greater* than it would be if the earth were a sphere of radius equal to that of the equator.

Actually this is found to be the case, and it is clear that by finding the actual discrepancy we can arrive at the relation between a and b.

The following table (in which the lengths of one degree of parallel are taken from Hinks' "Map Projections," and the spherical lengths are found by multiplying the equatorial length, or 69·17, by cosine of latitude) shows the kind of difference existing. All lengths are in miles for one degree of longitude.

Latitude.	Actual length.	Spherical length.
0°	69·17	69·17
10°	68·13	68·12
20°	65·03	65·00
30°	59·96	59·90
40°	53·06	52·99
50°	44·55	44·46
60°	34·67	34·59
70°	23·73	23·66
80°	12·05	12·01
90°	0·00	0·00

Other Methods.—In addition to the determination of measured arcs of latitude and longitude, the shape of the earth has been deduced by mathematical reasoning from theoretical considerations, first by Newton, also from the variations in the force of gravity at different places as indicated by the time required for a pendulum to swing, and from the parallax of the moon as observed in different latitudes. For particulars the reader is referred to the interesting article on the figure of the earth in the latest edition of the "Encyclopædia Britannica." Some results are given below.

Dimensions of the Earth.—According to the determination published by Captain Clarke in the *Philosophical Magazine* for August, 1878 ("Units and Physical Constants," by Professor Everett, chapter lxv.), the semi-axes of the ellipsoid which most nearly agree with the actual measurements of the earth were given in feet as follows:—

Greatest semi-axis major, or greatest semi-diameter at equator	20,926,629
Least ditto	20,925,105
Mean ditto	20,925,867
Semi-axis minor, or semi-polar diameter . .	20,854,477
Mean radius	20,890,172

The idea of supposing the equator to be elliptical is now generally discarded, partly on theoretical grounds, partly because it is not quite in agreement with the gravity measurements, and partly on account of the complication which results from it; nevertheless, it is true, as one would indeed naturally suppose (in consequence of the irregularities already referred to), that better agreement can be obtained, for any set of arc measurements, on such an assumption than by assuming that the earth is

a simple spheroid of revolution. Since 1878 many arcs have been measured.

If we take a = semi-major axis of the meridian ellipse, or radius of equator; b = semi-minor axis, or half the polar diameter, then the values as recomputed by A. R. Clarke in 1880, including the Indian arcs, were

$$a = 20{,}926{,}202 \text{ feet}$$
$$b = 20{,}854{,}895 \text{ ,,}$$

and if we put

$$\text{ellipticity} = \frac{a-b}{a} = e$$

then
$$e = \frac{1}{293 \cdot 5} \text{ nearly}$$

In 1841 Bessel found, by applying the method of least squares to ten measured arcs,

$$a = 6{,}377{,}397 \text{ international metres}$$
$$e = \frac{1}{299 \cdot 15}$$

and tables were constructed for these values.

According to the article in the "Encyclopædia Britannica" already mentioned, the latest results show that $e = \frac{1}{298 \cdot 3}$ about, and a may exceed 6,378,000 metres by about 200 metres.

The central bureau of the International Geodetic Association has however adopted, for practical reasons, the values for which tables exist, and have taken

$$e = \frac{1}{299 \cdot 15} \text{ after Bessel}$$

and
$$a = 6{,}377{,}397 \cdot 155(1 + 0 \cdot 0001) \text{ inter. metres}$$

The international metre = 3·2808257 feet. Hence the above values give—

$$a = 20{,}925{,}221 \text{ feet}$$
$$b = 20{,}855{,}272 \text{ ,,}$$
$$\text{mean} = 20{,}890{,}246 \text{ ,,}$$

This mean radius exceeds that given on p. 234 (namely, 20,889,000 feet) by 1246 feet, or about 1 part in 17,000. The value on p. 234 was taken from Rankine's "Civil Engineering" when the first edition of this work was in preparation, and it has not been considered necessary to re-work the examples.

Radii in Different Directions.—We have seen (p. 226) that

the radius of curvature along the meridian is given by the formula

$$\rho = \frac{a^2b^2 \sec^3 \lambda}{\{b^2 \tan^2 \lambda + a^2\}^{\frac{3}{2}}}$$

where λ is the latitude. In this put $b = a(1 - e)$, and we obtain

$$\rho = a\frac{(1 - e)^2 \sec^3 \lambda}{\{(1 - e)^2 \tan^2 \lambda + 1\}^{\frac{3}{2}}}$$

which easily reduces to

$$\rho = a\frac{(1 - e)^2}{\{1 - 2e \sin^2 \lambda + e^2 \sin^2 \lambda\}^{\frac{3}{2}}}$$

$$= b\frac{1 - e}{\{1 - 2e \sin^2 \lambda + e^2 \sin^2 \lambda\}^{\frac{3}{2}}}$$

Expanding this, and neglecting all powers of e higher than the first,

$$\underline{\rho = b(1 - e + 3e \sin^2 \lambda) \text{ very nearly}}$$

Now, if ρ_1 be the radius of curvature at a point B (Fig. 83, p. 223) in a direction at *right angles* to the meridian, this prime vertical circle will, for a short length, coincide with the parallel through B, and if two meridians be taken very near to that of B (but one on each side), the normals at the points where these meridians intersect the prime vertical will lie in the planes of the meridians, and hence will intersect at the point T in the polar axis NS.

This point is therefore the centre of curvature along the prime vertical; and BKE $= \lambda =$ latitude of B.

Hence $\qquad \rho_1 = \text{TB} = \text{BO} \sec \lambda = x \sec \lambda$

But $\qquad x^2 = \dfrac{a^4}{b^2 \tan^2 \lambda + a^2}$

Put $b = a(1 - e)$, and substitute, and we have

$$\rho_1{}^2 = x^2 \sec^2 \lambda = \frac{a^2 \sec^2 \lambda}{(1 - e)^2 \tan^2 \lambda + 1} = \frac{a^2}{1 - \sin^2 \lambda(2e - e^2)}$$

Hence, neglecting all powers of e higher than the first,

$$\rho_1 = a(1 - 2e \sin^2 \lambda)^{-\frac{1}{2}}$$
$$= b(1 - e)^{-1}(1 + e \sin^2 \lambda)$$

or $\qquad \underline{\rho_1 = b(1 + e + e \sin^2 \lambda)}$

This gives the radius of curvature along the prime vertical; and, except for very large surveys, it is generally sufficient for all purposes to assume that the area dealt with lies on the surface of a sphere whose radius is the mean between the radii of curvature along the meridian and along the prime vertical.

The value of this mean radius is clearly

$$\rho_m = b(1 + 2e \sin^2 \lambda)$$

For most purposes, however, the earth may be treated as a sphere having the following dimensions, derived from the figures on p. 228.

Length of an arc equal to radius or mean radius of the earth	=	20,890,172	feet	log 7·3199420
Length of arc subtending 1°	=	364,602	,,	,, 5·5618194
Length of arc subtending 1′	=	6076·7	,,	,, 3·7836681
Length of arc subtending 1″	=	101·28	,,	,, 2·0055168

As this mean radius differs by only about 70 feet from the more recently adopted value on p. 229, it has not been considered necessary to re-work any of the following exercises.

Great Circle, Azimuth, and Bearing.—In previous chapters the expressions "great circle," "azimuth," and "bearing" have been mentioned, and to some extent defined. Before, however, proceeding to the numerical consideration of the effect of the spherical form of the earth on surveys, it is well to recapitulate these definitions.

A "great circle" is any circle on a sphere whose plane contains the centre of the sphere, and consequently whose centre coincides with the centre of such sphere. It is, moreover, the largest circle that can be traced on the surface of a given sphere. The "equator" is a "great circle," as also are all "meridians." "Parallels of latitude" are not "great circles." A straight line as ranged out on the earth's surface by the surveyor is a portion of the "great circle." A "great circle" may be traced between two points on a terrestrial globe by stretching a cord between them, and an arc of a "great circle" is the shortest distance between any two points on the earth's surface. The "azimuth" of a straight line is the angle which it makes with the "great circle" of the "meridian" passing through a point in the same.

The "bearing" of a line is the angle which the plane of a "great circle" containing it makes with some standard "great circle," usually a meridian passing through the middle of the area to be surveyed. Thus, if A be a point on the standard meridian,

the "bearing" of another point B will be identical with the "azimuth" of the same point as determined from A. But though the "bearing" of A from B will be the arithmetical supplement of the bearing of B from A, the "azimuth" of A from B is not the supplement of that of B observed at A, but will differ by an amount depending upon the "difference of longitude" of the points at which the reciprocal "azimuths" are observed.

Thus, in Fig. 85, if P be the pole, the azimuth of B from A is the angle between the "meridian plane" PAO and the plane BAO of the "great circle" through A and B.

Similarly, the "azimuth" of A from B is the angle between the plane PBO and the plane ABO containing A and B.

Now if the meridian planes PAO and PBO were *parallel*, the angles they would make with the intersecting plane ABO would be equal or supplementary.

But as a matter of fact they *are not* parallel, but inclined to

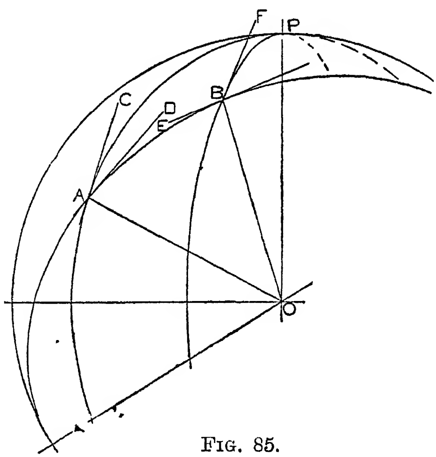


Fig. 85.

one another at an angle equal to the difference of longitude between A and B. Hence the reciprocal azimuths are not supplementary.

For "bearings," on the contrary, one "standard meridian plane" (passing preferably through the middle point of AB) is the "plane of reference," and the "bearing" of A from B or of B from A would be reckoned from planes both parallel to this standard plane of reference, hence the "bearings" *would be* supplementary.

Numerical Examples.—Let us consider the following case. From a point A, and *in any direction whatsoever*, let a line AB, Fig. 86, be ranged out of known length. At A and B respectively, let the lines AD, BC be exactly set out perpendicular to AB and prolonged to the points D and C, both at the same distance from AB. C and D are thus fixed with regard to AB, and three portions of great circles have been set out, each subtending known angles at the earth's centre, and AD and BC, moreover, cutting AB at right angles. If CD were ranged out as a straight line joining these points, it would be an arc of a great circle, but *not* cutting AD and BC at right angles, and *not* equal in length to the distance AB, as would be the case if the surface of the earth were a plane.

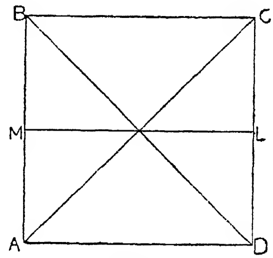


FIG. 86.

If the diagonals AC, BD be drawn, we obtain two right-angled spherical triangles ABC, BAD, equal in all respects.

It is required to calculate all the parts of these triangles, as well as the length of CD, the angles BCD and ADC, and the middle height LM, and to compare with the corresponding values for a plane figure.

Put AB $= \theta =$ angle subtended by AB at the centre of the sphere.

BC $=$ AD $= \phi$.

Then the right-angled spherical triangle ABC can be solved by the formulæ on p. 35, from the following parts :—

$$B = 90^\circ \qquad BC = \phi \qquad AB = \theta$$

This is left as an exercise to the student.

The results are—

$$\cos AC = \cos \theta \, . \cos \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$\tan BAC = \sin \theta \, . \cot \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$\tan ACB = \sin \phi \, . \cot \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To find the remaining parts refer to Fig. 87, and suppose R is the pole of the great circle AMB (that is, BC and AD produced meet at R, and each is a complete quadrant). We proceed to solve the triangle LDR.

We have seen that AR $=$ MR $= 90^\circ$,

$$\therefore DR = 90^\circ - \phi$$

Also angle RLD $= 90^\circ$, because the triangle RCD is isosceles, and L is the middle of the base; and the angle DRL $= \frac{1}{2}\theta$, because the angle DRC is the angle between the planes of DR and BR, and as AR and BR are both quadrants this is measured by the side AB, or θ.

Now put ML $= \mu$,
and CD $= x$.

$$\therefore RL = 90^\circ - \mu$$

and
$$DL = \frac{x}{2}$$

Taking L $= 90^\circ$, therefore, in the triangle RLD, the five parts for Napier's rules are

$$RL = 90^\circ - \mu \qquad LD = \frac{x}{2}$$
$$90^\circ - LDR \qquad 90^\circ - DR = \phi$$
$$\text{and } 90^\circ - DRL = 90^\circ - \frac{\theta}{2}$$

The solution gives

$$\sin \frac{x}{2} = \sin \frac{\theta}{2} \cdot \cos \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\tan \mu = \sec \frac{\theta}{2} \cdot \tan \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{and } \tan \text{LDR} = \cot \frac{\theta}{2} \cdot \operatorname{cosec} \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

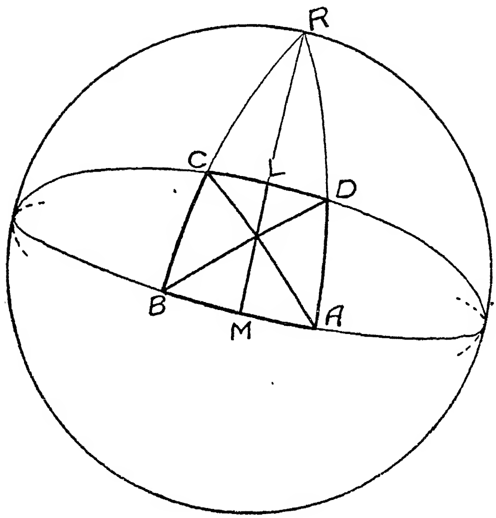


Fig. 87.

The student should check these as an exercise.

Remembering that ADC = 180° − LDR
and that by symmetry DCB = ADC
DBA = BAC
and ADB = ACB

it is clear that we can now find all the parts in any given numerical example.

We shall suppose that BC, AB, and AD are each 60 nautical miles in length, that is, that each one subtends 1° at the centre of the sphere which most nearly agrees with the earth at the centre of the given area, and we shall assume a radius for this sphere of 20,889,000 feet, corresponding to the mean radius for a latitude of about 30°.

From equation (4),

$$\sin \frac{x}{2} = \sin 30' \cdot \cos 1^\circ$$

whence we obtain

$$x = 59' \, 59{\cdot}452'' \\ = 3599{\cdot}452''$$

Multiply this by the number of feet in one second, *i.e.* by

$$\frac{\pi \times 20{,}889{,}000}{180 \times 60 \times 60}$$

and the length of the arc CD becomes	364,526 feet
Whereas if the earth's surface were flat, it would be equal to AB, or 3600 × feet in one sec. . . .	364,582 ,,
Hence the value of CD, if calculated by plane geometry, would be too long by the difference, or . . .	56 ,,

Thus, by neglecting the spherical shape of the earth an error is made in the length of CD of about 56 feet, or an average rate of nearly 0·93 foot per nautical mile.

Again, from equation (5)

$$\tan \mu = \tan 1^\circ \sec 30'$$

whence

$$\mu = 1^\circ\ 0'\ 0{\cdot}138''$$
$$= 3600{\cdot}138''$$

Reducing to feet as before,

Arc LM . .	364,596 feet
By plane geometry	364,582 ,,
Showing a deficit of	14 ,,

Hence the arc LM calculated by plane geometry would be 14 feet *too short*, being an average rate of about 0·23 of a foot per mile.

Similarly, the diagonals and the angles can be calculated and compared. The student should work these out for himself.

The following table gives the results of the computation showing excess or defect due to calculation by plane trigonometry.

Excess or Defect by Plane Trigonometry.

	Plane.	Spherical.	
AB = BC = AD	364,582	364,582	
Side CD	364,582	364,526	+ 56 feet
Diagonals AC, BD	515,596	515,583	+ 13 ,,
Middle distance LM	364,582	364,596	− 14 ,,
Angles ABC, BAD	90° 0′ 0″	90° 0′ 0″	—
,, ABD, BDA, ACD, BCD.	45° 0′ 0″	45° 0′ 16″	− 16 secs.
,, CAD, DCA	45° 0′ 0″	44° 59′ 44″	+ 16 ,,
Sum of three angles of any one of triangles ABD, etc.	180° 0′ 0″	180° 0′ 32″	− 32 ,,
Sum of angles of rectangle ABCD	360° 0′ 0″	360° 1′ 4″	− 64 ,,

It will be seen that the differences are trivial, even in so large an area as 60 by 60 square (nautical) miles, and are smaller than the probable error of measurement. They would also be inappreciable on paper, if the area were plotted to any reasonable scale. Even if a scale large enough to make these differences appreciable were adopted, the several lines, as calculated by spherical trigonometry, could not be laid down so as to agree with each other in every respect.

Spherical Excess of Triangle.—In the above example it will be observed that the sum of three angles of any of the spherical triangles into which the figure ABCD is divided exceeds two right angles by a certain number of seconds.

This is due to the spherical excess already referred to (p. 38).

It is found by the following rule :—

Let A = area of triangle in square feet,
r = radius of sphere in feet,
E = spherical excess in seconds of arc.

$$\therefore E = \frac{A \times 180^\circ}{\pi r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

A simple formula may be deduced to compute the spherical excess of any triangle from its area in square miles, acres, feet, or any other units.

The radius of the perfect sphere which has been taken as most nearly approximating the form of the earth, is about 20,889,000 feet.

Sum = log r^2 { log r in feet	7·3199176
{ log r	7·3199176
Log π	0·4971499
Log πr^2	15·1369851
Colog πr^2 (deducting from zero)	$\overline{16}$·8630149
Log 180° in seconds	5·8115750
$\therefore$ Constant log or log $\frac{E}{A}$ (for sq. ft.) is	$\overline{10}$·6745899

For example, the area of the triangle ABC, Fig. 86, is

$$\frac{364{,}582 \text{ ft.} \times 364{,}582}{2} \text{ sq. feet}$$

Colog 2 = log 0·5	$\overline{1}$·6989700
Log 364,582	5·5617950
Ditto for square	5·5617950
Constant log as above	$\overline{10}$·6745899
$\therefore$ Spherical excess of this triangle = 31·4 seconds	1·4971499

It is perhaps more convenient to have the constant for areas in square miles.

Log constant for feet	$\overline{10}$·6745899
Log 5280	3·7226339
Ditto for square	3·7226339
Constant log or log E for 1 square mile	$\overline{2}$·1198577

Antilog = 0·0132″

From the above it will be seen that the spherical excess of a triangle having an area of one square mile is less than one-seventieth of a *second*—a *negligible* quantity in ordinary surveying operations.

The spherical excess of the figure which has been dealt with, viz. 60 square (nautical) miles in area, amounts to 64 seconds. If, instead of a square, a polygon of equal area had been surveyed by traversing, there would have been a closing error in the angles amounting to 1′ 4″, even supposing absolute accuracy in the work. As the perimeter would be about 240 miles, the probable number of sides would not be less than 240. In this case the spherical excess would amount to about one-quarter of a second per side.

This is vastly less than the probable error of an angular observation of any ordinary theodolite.

Convergence.—We have seen that $\sin\frac{x}{2} = \sin\frac{\theta}{2}\cos\phi$, where x and θ are the angular measures of CD and AB (Fig. 88) respectively.

Now if these angles are less than, say, 2°, we may take the arcs as proportional to the sines of the subtended angles without sensible error.

Hence, for areas less than 2° square

$$\text{CD} = \text{AB}\cos\phi$$

$$\therefore \text{AB} - \text{CD} = \text{AB}(1 - \cos\phi)$$

or

$$\frac{\text{AB} - \text{CD}}{\text{AB}} = 1 - \cos\phi$$

$$= 2\sin^2\frac{\phi}{2}$$

This formula gives the *rate of convergence* of the planes or lines BC, AD.

If we multiply by the number of feet in one nautical mile (=6076) we obtain the result in feet per nautical mile of AB.

Again, if ϕ be less than 2°, we may write $60 \times \frac{\phi}{2} \times \sin 1''$ instead of $\sin\frac{\phi}{2}$, where ϕ is in minutes.

$$\therefore \frac{\text{AB} - \text{CD}}{\text{AB}} = 6076 \times 2 \times 3600 \times \sin^2 1'' \times \tfrac{1}{4}\phi^2$$

$$= 0{\cdot}000257\phi^2$$

This formula therefore gives the rate of convergence, in feet per nautical mile of AB, of two straight lines, *each perpendicular to AB to start with*, where ϕ is the spherical measure of each of these perpendiculars expressed in minutes, and where neither AB nor the perpendicular is much greater than 120 nautical miles in length. (*Note* 1° = 60 n. miles.)

As we have seen, it amounts to 0·93 foot when $\phi = 1°$ = 60 minutes. For other small values of ϕ it is proportional to ϕ^2.

Thus if the rectangle ABCD (Fig. 86, p. 233) already dealt

with were referred to a meridian POQ (Fig. 88) running through the centre thereof, and to a base line ROS, and treated as a plane figure in calculating co-ordinates, then the rate of error in CS, SD, AR, and RB would be *one-fourth only* of the rate of error in CD when referred to AB as axis. That is, in the case assumed, the rate of error would be reduced from 0·93 foot per mile to about 0·23.

Up to this point, it is clear that the error, either in computation or in plotting, due to the assumption that the area surveyed is a plane and not a portion of a sphere, is wholly negligible in

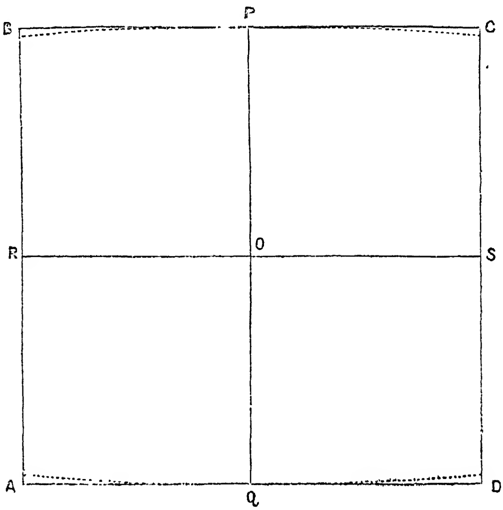


FIG. 88.

comparison with the unavoidable errors of any ordinary method of surveying.

Convergence of Meridians.—The last point to be considered is the convergence of the meridians.

If BC (Fig. 89) be a true north-and-south line, and AD be a line which would be parallel to BC if the figure were on a plane surface, then to what extent does AD differ from the true north-and-south line at A in direction?

We shall assume, for the sake of argument, that BA is at right angles to BC (that is, that it is due east at B). Hence AD must be perpendicular to AB at A, so that for a plane figure it would be parallel to BC.

That is, AD produced passes through the pole Q of the great circle AB, and BAD = 90°.

Now let P be the terrestrial pole, say the north pole.

Then AP is the direction of the meridian (*i.e.* the true north-and-south line) at A, and we require the angle PAQ, by which this differs from AQ.

Let λ = latitude of B = EOB in the figure, where EF is the equator.

$$\therefore \text{BOP} = 90° - \lambda$$

And put AOB = ϕ = spherical measure of AB.

Therefore in the triangle APB

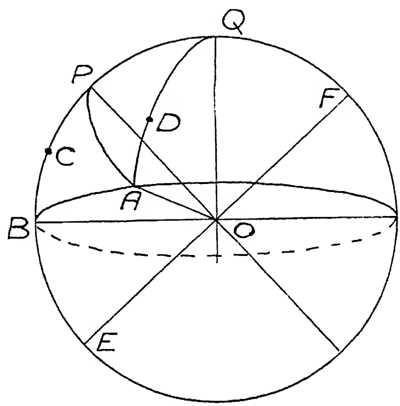


FIG. 89.

$$\text{B} = 90° \quad \text{BP} = 90° - \lambda \quad \text{BA} = \phi$$

Solving by the rules on p. 35

$$\cot \text{BAP} = \sin\phi \tan\lambda$$

But the required convergence = QAP = 90° − BAP.
Hence we have the rule

$$\textit{tan convergence} = \sin\phi \,.\, \tan\lambda$$

where λ is the latitude and ϕ the spherical measure of the distance AB, which is the *departure*.

If AB = 60 nautical miles, or $\phi = 1°$ and latitude of B = 45° = λ,

we have
$$\text{tan convergence} = \sin 1° \tan 45°$$
$$\therefore \text{convergence} = 59' \, 59{\cdot}5'' \text{ nearly}$$

Now it must be observed that this angle is far greater than the probable error in the determination of the *true* north with a common theodolite. If B were in a lower latitude, say 15°, we should have less convergence, but still an appreciable amount, namely, 16′ 4″.

When the departure is not great, say under 60 miles, this expression may be simplified, so that the convergence of the meridians may be calculated, without the reduction of length to angle.

For, from the above formula we have

$$\tan \text{convergence} = \tan \text{latitude} \times \sin \phi$$

but for small angles, the sines, tangents, and circular measure, *i.e.* $\frac{\text{arc}}{\text{radius}}$, may be taken as equal.

Hence $\sin \phi = \frac{\text{departure}}{\text{radius of earth}}$, if ϕ be small

$$\begin{aligned} \text{and } \tan \text{convergence} &= \text{circular measure of convergence} \\ &= \text{convergence in minutes} \times \text{circular measure of } 1' \\ \therefore \text{convergence in minutes} &= \frac{\tan \text{convergence}}{\tan 1'} \\ &= \frac{\tan \text{latitude} \times \sin \phi}{\tan 1'} \\ &= \frac{\tan \text{latitude} \times \text{departure}}{\text{radius of earth} \times \tan 1'} \end{aligned}$$

The "departure" and the "radius" must be in the same units. When these units are fixed, the logarithms of $\frac{1}{\tan 1' \times \text{radius of earth}}$ is constant, and its value can be calculated.

Thus, for departure in nautical miles, constant log $= 0$
" " statute " " $= \bar{1}\cdot 9390$
" " chains of 66 feet " $= \bar{2}\cdot 0359$
" " feet " $= \bar{4}\cdot 2164$

Example.—Departure = 60 nautical miles, latitude 45° :

$$\begin{aligned} \text{constant log} &= 0\cdot 0000 \\ \log \tan 45^\circ &= 0\cdot 0000 \\ \log 60 &= 1\cdot 7782 \\ \hline & 1\cdot 7782 = \log 60 \end{aligned}$$

$$\therefore \text{convergence} = 60'$$

With a "departure" of 12 statute miles in latitude 50° :

$$\begin{aligned} \text{constant log} &= \bar{1}\cdot 9390 \\ \log \tan 50^\circ &= \cdot 0762 \\ \log 12 &= 1\cdot 0792 \\ \hline & 1\cdot 0944 = \log 12\cdot 43 \end{aligned}$$

$$\therefore \text{convergence} = 12'\cdot 43$$

This convergence of meridians has no connection with any distortion caused by neglecting the spherical form of the earth. It is an *actual fact*, and would be observed if the theodolite were set up and the true "north-and-south" line were determined. Thus, referring to Fig. 89, assume as before that BC is a part of a meridian, and the angle ABC a right angle, then, if the *true* north were determined by a theodolite set up at A, the true "north line"

found in the latitude of 45° north would not be parallel to BC, or at right angles to AB, but would make an angle of 89° with it, if AB = 60 nautical miles.

The "convergence of the meridian" can be observed in any map of an extended area of the globe's surface, and even in the relatively small area which is represented by a sheet of the Ordnance Survey of England, on the scale of 6 inches to one mile. In these maps the degrees, minutes, and seconds of longitude are figured both on the upper and lower margins. It will be noticed that there are a greater number of degrees, minutes, and seconds marked along the upper margin of the paper than along the bottom. The true "north-and-south" line through any point is, therefore, not a line parallel to the lateral margins, but one drawn through the point and intercepting equal "longitudes" at the top and bottom of the sheet.

The above considerations show that *the error due to the neglect of the "spherical form of the earth,"* as regards the computation of distances and the projection of points, is *negligible* as compared with the inevitable errors of ordinary means of measurement in the field, and of projection on paper. Even if the survey were made with the most accurate methods known to science, computing "latitudes" and "departures" by spherical geometry, the points so obtained could not be projected upon the paper without distortion in one direction or the other.

On the other hand, *the errors due to the neglect of the "convergence of meridians"* must not be disregarded, for this is an actual fact, both on the surface of the earth and on the plan or map.

Examples on Convergence of Meridians, Bearing, and Azimuth.—The case of setting out a parallel of latitude as a boundary between two states affords an instructive example of the difference between "bearing" and "azimuth" and of "convergence of the meridians."

Suppose that it be desired to set out the parallel of latitude 54° north, a place in exactly that "latitude" being given as a starting-point. It is desired to range out a straight line from the starting-point in such direction that the extremity thereof shall also be in latitude 54°. Required the "azimuth" of the line at the starting-point, its intended length being 364,580 feet (60 nautical miles nearly). Now this distance subtends 1° at the centre of the earth. In Fig. 90 let P be the pole, A and B the terminal stations. We have the spherical triangle PAB in which the arcs PA and PB = 90° − latitude, or

$$\text{co-latitude} = 36^\circ\ 0'\ 0''$$
$$\text{and arc AB} = 1^\circ\ 0'\ 0''$$

From these data the angles at A and B may be calculated by the rules for right-angled spherical triangles, as follows. Bisect the arc AB in C. Draw a great circle through PC cutting AB in C. Then PAC is a right-angled triangle, in which C is the right angle, and in which the hypotenuse c and the side p are given and the angle A is sought.

The formula is

$$\cos A = \cot c \tan p$$

	Log
$c = 36^\circ\ 0'\ 0''$, log cot $36^\circ\ 0'\ 0''$	10·1387390
$p = \quad 30'\ 0''$, log tan $\quad 30'\ 0''$	7·9408584
$\therefore$ log cos A = log cos $89^\circ\ 18'\ 42''$	8·0795974

and $A = 89^\circ\ 18'\ 42''$

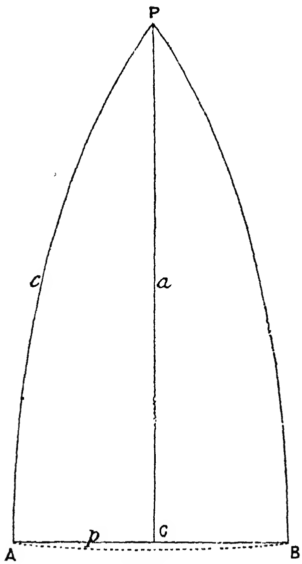


FIG. 90.

Hence the surveyor would range out a straight line ABC, commencing with an "azimuth" 89° 18′ 42″ north-east and prolonging the same on the ground until he had chained 364,580 feet, when astronomical observation would show that he was again in latitude 54°, and that the "azimuth" of the line was 89° 18′ 42″ north-west. This line is, of course, really a portion of a great circle, but appears straight on the ground, and is shown straight on the figure.

If he paused exactly halfway and checked his latitude, he would find that he was not in "latitude" 54°, but in a higher "latitude." The "latitude" of the point C, the middle point, may be calculated from the same right-angled triangle, the side a being the co-latitude of the point C.

The formula is

$$\cos a = \frac{\cos c}{\cos p}$$

log cos c = log cos $36^\circ\ 0'\ 0''$	9·9079576
log cos p = log cos $\quad 30'\ 0''$	9·9999835
$\therefore$ log cos a = log cos $35^\circ\ 59'\ 49''$	9·9079741

and $a = 35^\circ\ 59'\ 49''$

$\therefore$ latitude of point C = $54^\circ\ 0'\ 11''$

Therefore the point C would be 11″ north of the true parallel of "latitude," or with the mean radius of the earth 1114 feet nearly. This offset, being set off due south at C, would fix the middle point of the arc of the desired parallel of latitude, which is represented by the dotted line in the figure. Similarly, the offsets of other points may be calculated and set off, so as to fix the curved line representing the "parallel of latitude."

The same curve might also be traced by continually setting out *very short* lengths of line, always with an azimuth of 90° east. The total distance chained would be found to be greater than in the first instance.

It is instructive to calculate the "difference in length" between the "great circle" and the "parallel of latitude" between A and B.

The first step is to calculate the "difference in longitude" between A and B. This is the angle at the pole of the earth, APB, which again is twice the angle APC in the right-angled spherical triangle APC.

$$\cot P = \cot p \sin a$$

log cot p = log cot	34′ 0″	12·0591416
log sin a = log sin	35° 59′ 49″	9·7691868
log cot P = log cot	51′ 2·5″	11·8283284
multiply by 2 .	2	

6125″ = 1° 42′ 5″ = diff. of longitude between A and B.

The length of the arc of a small circle being proportional to the radius thereof, or to the cosine of the "latitude," it is merely necessary to multiply together the number of feet in a second of arc of a great circle, the number of seconds of "longitude" and the cosine of the "latitude."

log feet in one second	2·0054925
log 6125 seconds	3·7871061
log cos 54° 0′ 0″	9·7692187
∴ log length of arc of small circle or parallel of latitude = log 364,600 feet .	5·5618137

Now, 364,582 feet is the length of arc of great circle,

∴ difference = 18 feet

If, on the other hand, the surveyor proceeded to lay out a line from A, making each short length run exactly east (by means of a perfectly corrected compass, for example), he would reach B and lay out a parallel of latitude correctly. But if, confounding "azi-

muth" with "bearing," he proceeded to plot the line as straight, he would delineate a curve as a straight line, and place the middle point about 1114 feet to the north of its proper position. He would obtain also a slightly erroneous distance between A and B.

If, on the other hand, the great circle from A to B were plotted at right angles to the "north-and-south" line through the middle point C (as in the figure), the line on the map would be straight, as it would appear on the earth, whilst the points on the "parallel of latitude" would appear to be on a curve, as they would actually be on the ground.

To show the "azimuth" of the line at any point on its length, it would only be necessary to draw the "meridian" with the "convergence" proper to the point. Thus the "meridians" at the extremities would make angles of 89° 18′ 42″ with the line, or in other words, the "convergence" of the "extreme meridians" would each be 41′ 18″ with the central "prime meridian." The several points at the same "latitude" would be obtained correctly from a curved line cutting each successive meridian at right angles. The "parallel" of 54° would be none other than the curve set out by offsets from the great circle, as already described, and measured from this the "latitude" of the middle or any other point would be 54.

Suppose, again, the case of a survey for a railway in latitude of 60° north, such that the "departure" between its extremities is 20 miles.

The surveyor begins by determining astronomically the "azimuth" of the first line from the starting-point, and takes that as its "bearing." He measures the various angles along the line, and from them calculates the "bearing" of the last line.

He wishes to check his angular measurements, and at the finishing-point determines the "azimuth" of the last line, and compares it with the computed "bearing."

If his angular measurements and his astronomical observations were absolutely perfect, he would, nevertheless, find a difference between the computed "*bearing*" of the last line and its "*azimuth*," as observed astronomically. This would be due to the convergence of the meridians, which may be thus calculated by the approximate formula—

constant log (p. 240)	$\bar{1}\cdot9390$
log tan 60°	$10\cdot2386$
log 20	$1\cdot3010$
∴ log convergence = log 30·10 . .	$1\cdot4786$
and convergence = 30′·10	

Now, if he were to neglect this "convergence," and adjust his angles so as to make the final "bearing" of the last line agree with the observed "*azimuth*," at the final station, he would distort his line by half a degree, or if there were twenty lines, the angles would be altered by about $1\frac{1}{2}$ minutes each, a quantity quite within the limits of accuracy of an ordinary theodolite.

Again, if he were to start to survey a further section, using an "azimuth" observed at the end of the first section for "bearing" of the first line of the second section, then the relative directions of the two sections would be distorted by half a degree.

Therefore from these examples it will be seen that, as above stated, the "convergence of the meridians" must not be neglected in checking angular work by astronomical observations.

It is, moreover, seen that even in so short a distance as 1 mile, at latitude 60°, the convergence amounts to $1\frac{1}{2}$ minutes, a quantity in excess of the probable error of an ordinary "azimuth" observation.

General Conclusions.—The general conclusions at which we arrive therefore are, that if we start at a given point A (Fig. 87) work, say by traversing, take as "standard direction" some line making a known angle with AB at A, and assume that all straight lines which make this same angle with AB (or AB produced) are parallel to one another (in other words, that the standard direction remains parallel to itself), then we are plotting the figure as if it were on a plane surface, and the error we thereby introduce is less than the unavoidable errors of measurement and plotting by any ordinary methods of work, if the area be no greater than about 120 nautical miles square (= 20,000 square statute miles about).

But if the original standard direction were a true north-and-south line; it must be remembered that north-and-south lines do *not*, in general, make a constant angle with any given straight line, but converge more and more rapidly as the latitude increases.

Hence the standard direction would not *continue* to be a north-and south line, and if in plotting we took the vertical on paper as the standard (or north) direction at A, then, when the drawing was complete, if we wished to show the direction of north at any other point, we could not draw it as a vertical line on paper. The error in doing so would be appreciable even in a distance of 1 mile in high latitudes; and we must allow for the convergence of the meridians between the point A and the point considered.

Thus suppose our survey were made without any errors of measurement at all; that at A we find the exact latitude and longitude astronomically, as well as the azimuth of AC.

Then, taking true north at A as standard direction, from the

included angles and lengths we calculate differences of latitude and departures, as described in Part I., until we arrive at C (Fig. 87).

From the linear difference of latitude A to C and the known latitude of A we can calculate the latitude of C.

Thus if L = linear diff. of lat.,

ρ = radius of earth's curvature at centre of area,

$$\text{diff. of lat.} = \frac{L}{\rho} \text{ radians}$$

$$= \frac{L}{\rho \times \sin 1''} \text{ seconds}$$

Similarly, if D = total departure,

$$\text{diff. of long.} = \frac{D}{\rho \cos \lambda \sin 1''} \text{ seconds}$$

where λ is the latitude of the origin A. Here we are treating the figure as plane, and merely reducing the linear distances to angles at the finish. A rather better result for diff. of longitude may be obtained by taking the *mean* value of the latitude instead of λ.

Finally, from the traverse observations we can find the bearing of CA from C.

This last would differ from the original bearing or azimuth of AC at A $\pm 180°$ by an amount less than what the actual errors of measurement would be in any ordinary surveying work; and if we could again find the latitude and longitude of C without error, they would agree with those derived as above from the initial values at A within the same amounts of error.

But if we also find the *azimuth* of CA at C this will be from true north at C, which is *not parallel* to our standard direction, and we should expect to find that this differed from the azimuth of AC at A $\pm 180°$ by an amount greater than the allowable errors in good work.

Before we could use it as a check on the work we must allow for the convergence of meridians as above.

Finally, if we were to take our original origin and standard direction through O, the middle of the area, then the distances to the extremities would be only half as great, and the error in latitude, bearing, etc., due to neglecting the spherical shape on the final check at any corner of the figure, would be reduced to about *one-quarter* (p. 238).

Big Areas.—For very big areas the object is that all points should be shown with their correct latitudes and longitudes.

This is actually impossible to any definite scale on any plane

maps, as it could only be done if it were possible to develop the surface of a spheroid.

Hence some method of map projection is chosen (see Chap. X.) according to the shape, area, and position of the ground and the purpose of the map.

A "graticule" or network of lines of latitude and longitude is then laid down accordingly. On this, triangulation stations, etc., are shown, plotted by their latitudes and longitudes, observed or calculated, and the rest of the work adjusted to these by methods already described.

CHAPTER VII

TIDAL PHENOMENA

Object of Chapter.—The object of the present chapter is to instruct the surveyor as to the general character of the phenomena which he may expect to observe in the course of conducting Marine Snrveys, and as to the nature of the observations which must be made in order to determine Mean Sea-level, as well as for the determination of the constants required for the construction of tide-tables, giving predictions of the times and heights of high and low water, respectively, day by day. Space will not permit of giving the details of the methods of deducing constants from observations taken, or of the manner of using them for predictions, these operations being too complicated and laborious for the surveyor to undertake.

The writer here begs to acknowledge the valuable assistance which he has received in the preparation of this chapter from Mr. E. Roberts, of the "Nautical Almanac" Office, who prepares the tide-tables for Indian and other ports. The tide-tables of Hong-Kong are based on observations made in the first instance by the writer.

Recurrence of High and Low Water.—It is an observed fact that the surface of the sea rises and falls in approximate accordance with the apparent movement of the moon, high water and low water recurring twice daily (approximately), at some fixed interval of time after the moon's transit. Currents also obtain whose direction and velocity vary in accordance with the same phenomenon. For the present, attention will be confined to the rise and fall of the surface-levels.

The *range* of a tide is the difference between the water-levels at high and low water respectively. The half-range is called the *amplitude*.

The greater tides, occurring at full and new moon, are known as *Spring tides*, the smaller, occurring at the first and last quarters, are called *Neap tides*. The time-interval between the moon's transit and the instant of high and low water, lengthens and shortens

during a lunar month. This is known as the *priming* and *lagging* of the tide.

On Admiralty Charts the words "High-water at full and change . . . hours" occur. This is the hour at which high water occurs at full and new moon (the days when the moon transits at noon and midnight respectively), so that the numbers given are, the hours (afternoon and forenoon) of high water on the days of new and full moon respectively. Since the moon makes a complete circuit round the earth in about twenty-eight days, high water will occur, roughly, fifty minutes later for each day of the moon's age.

Tidal Establishment of a Place.—The hour of high water "full" and "change" is called the *Establishment* of the place. Suppose that the Establishment of a place is six hours, then at new moon the time of high water is at 6 a.m., followed by a second high water at about 6.25 p.m. on the same day. On the next day the high water will occur at 6.50 a.m. and 7.15 p.m., and so on till full moon. The "Establishment" therefore affords a means of *roughly* determining the time of high water on any given day of the month, by adding fifty minutes for each day of the moon's age. This method is not accurate, though possibly near enough for purposes of navigation. In the case of important ports, "tide tables" are published giving the time of high water day by day.

Statistical Theory of Tidal Movements.—The theory of "tidal movements" is extremely complex, though, generally speaking, they

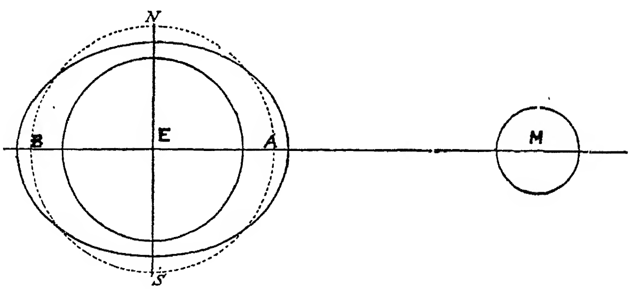


Fig. 91.

are undoubtedly due to the attraction of the sun and moon drawing up the water. An explanation of observed tidal phenomena (as regards their general character), may be obtained, by assuming that the globe is entirely covered with a uniform layer of water, and that the sun and moon move intermittently and not uniformly

in their courses, pausing in successive positions for a sufficiently long time to allow the layer of water to assume the form due to their attractions. This assumption is known as the "Statical theory of tides," and was first propounded by Newton.

Let E be the earth and M an attracting body, such as the moon (*vide* Fig. 91).

Moon's Effect on Tides.—If there were no attracting body in the neighbourhood of the earth, the water would form a uniform layer round the solid spherical kernel, as shown in the dotted lines, Now, suppose that the moon M came into existence, it would exert an attraction upon the particles of the solid earth, and of the water surrounding it. By the law of gravitation, the attraction of the moon upon any particle of matter is inversely as the square of the distance between the moon's centre and the particle in question. Consequently, the particles of water at A will be more strongly attracted than those of the solid earth, whose centre of gravity is at E, and these again will be more strongly attracted than those of the remote water particles at B. The water-surface will therefore assume the form of a spheroid, with its longer axis pointing towards the attracting body. This form will follow the moon in its *apparent* daily course, and the result will be that twice in every lunar day there will be two "high waters" and two "low waters" at every place. This conclusion is in general accordance with experience, though the instant of high water does not coincide with the moon's transit, but usually occurs somewhat later. This is no doubt due to the fact that the earth is not wholly covered by water. The fact that there are two "high" and two "low waters" in a lunar day is thus explained.

Sun's Effect on Tides.—The sun also is an attracting body, and produces similar effects.

Effect of Sun's and Moon's Attractions combined.—Hence, two high and two low waters in a solar day (24 hours) would take place were the moon absent. Both sun and moon being present, tides would be produced which would be the resultant of their two attractions.

Neaps and Spring Tides explained.—Now the moon in its apparent course lags (on the average about fifty minutes daily) behind the sun. Consequently, the moon changes its position in the heavens with regard to the sun, making a complete tour of the heavens in a lunar month. At new moon, the sun and moon are in line and on the same side of the earth. The two attractions coincide in direction, and a tide equal to the sum of the lunar and solar tides may be anticipated, *i.e.* a spring tide. So also at full moon when the moon is directly opposite to the sun. When the

moon is at its quarters, or at right angles to the sun, its attraction is exerted on points of the sphere where that of the sun is producing a depression. The resultant tide is, therefore, the difference of the solar and lunar tides, *i.e.* a neap tide. This agrees with experience, though the highest tides do not, at all places, coincide with the instants of new and full moon, but often are some days later. The phenomena of neaps and springs are therefore explained.

The Terms Priming and Lagging explained.—The phenomena of priming and lagging may be explained by similar reasoning. The lunar and solar tides combine and form a resultant ellipsoid, or pair of tides, whose longer axis will be in the line of the resultant of the pair of attracting forces. Taking the transit of the moon as the regulating phenomenon of the tide, because its attracting force is greater than that of the sun, then it is evident from Fig. 92 that at new and full moon both bodies will be on the meridian of the place of observation at the same instant, and that the resultant attraction will be the sum of the two forces acting in one straight line and passing through the centres of the three bodies. High water will occur at the instant of the moon's transit, and again twelve lunar hours later. At the first and last quarters, the tidal axis will follow the line of the resultant of the two attractions.

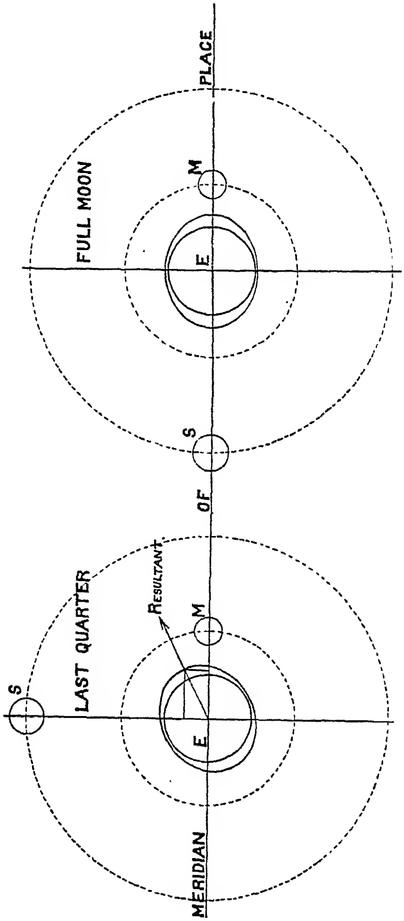


Fig. 92.

At the first quarter, the resultant will be in advance of the moon, so that high water may be expected to occur *before* the moon's culmination, whilst at the last quarter it will occur *after* her culmination.

Effect of Change of Declination and Right Ascension of Sun and Moon.—Hitherto we have considered the effects of the sun's and moon's attraction, as though they acted in the plane of the earth's equator. But neither the sun nor the moon are at all times situated in the plane of the equator. The moon changes her declination (sometimes being as much as $21\frac{1}{3}°$ north, and at others as much south of the equator), the variation taking place in a lunar month. The sun also varies its declination $23\frac{1}{2}°$ from north to south in the course of a year.

Now it is obvious that when the sun and moon have nearly the same declination, and are on the meridian, the resultant attraction will be nearly equal to the sum of the two attractions. When, however, they have widely different directions, the resultant will be less than the sum. This is shown on the right-hand side of Fig. 93. A variation in tidal ranges may therefore be expected to follow the variations in declination, as well as in right ascension, of both the sun and the moon, the period of the latter being a *lunar month* and of the former *a year*.

Diurnal Inequality.—The difference in the height of successive tides is called the *diurnal inequality*, which in British waters is but

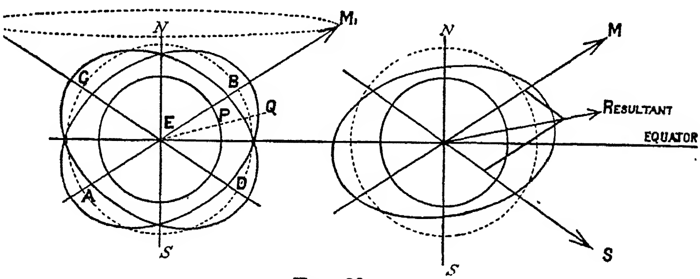


FIG. 93.

small, though very large in other places, such as Aden, where it is sometimes so large that one tide of the day entirely disappears, being only represented by a *slowing down* in the *rate* of the rising and falling.

Variation in declination serves to account for this phenomenon, and is explained as follows.

Let us consider the moon in the first instance (*vide* Fig. 93, left side).

Let EM_1 be the direction of the moon's attraction, when it passes the meridian of some point P not in the equator. The longer axis of the watery spheroid will be in the direction $AEBM_1$, and the radius of the spheroid drawn through P will be EPQ. Twelve lunar hours later the moon will be again in the meridian of P. The longer axis of the spheroid will then be in the direction of DCM_2, and its radius in the direction of EP will be manifestly shorter than EPQ. The fact of alternate large and small tides, known as the "diurnal inequality," is thus explained.

Effect of the Relative Distance of Sun and Moon on Tides.—Now, assuming the attractive forces of the sun and the moon to be respectively *directly* as their masses and *inversely* as the squares of their distances from the earth, the sun's attraction will be to that of the moon as 1 is to 2·1. The range of spring tides might therefore be expected to bear to that of neaps a ratio of $2{\cdot}1 + 1 = 3{\cdot}1$ to $2{\cdot}1 - 1 = 1{\cdot}1$, though so great a difference is rarely observed.

Tidal Perturbations.—Lastly, tidal perturbations exist, which are due to the *variations* in the distances of the sun and moon, and to the variations in their rates of apparent motion.

From the above it will be seen that the theory of tides is sufficiently complicated, even on the comparatively simple supposition of an uninterrupted and completely enveloping ocean. The fact of land and water intervening renders these problems even more so.

Theoretical considerations, based on the assumption of an all-enveloping ocean, merely indicate the kind of phenomena that may be expected at any point on the earth, but the amount of the influence of each set of conditions, and the time at which it will be felt, can only be determined by actual experiment at each particular spot. In short, theory only indicates what is sought for, but not the extent of the result.

The range of tides is very variable. Judging from observations taken at small islands rising abruptly from very deep water, such as Mauritius or the West Indian Islands, the maximum range in mid-ocean is but from four to five feet. But when the wave rolls up narrow channels (especially if trumpet-shaped, the range increases greatly, and may be sometimes as great as 80 feet.

Nature and Extent of Tidal Observations.—To obtain reliable information as to tidal phenomena, it is necessary to take hourly readings of the sea's level above some datum-plane. These must be continued night and day for 369 days 3 hours. A mere record of the *times* and *heights* of high and low water will not suffice;

indeed, such observations will probably give very erroneous results. Even mean sea-level cannot be obtained accurately in this manner. Neglect to take continuous observations is the principal cause of tidal phenomena being pronounced to be anomalous.

An approximation to mean sea-level may be obtained by observations extending over a lunar month, but absolute accuracy can only be arrived at by observations extending over the period stated above.

Assumptions on which Tides are Investigated and Predicted.—Guided by the general principles which have been briefly described, and having to hand the results of *continuous* observations, extending over 369 days 3 hours (25 lunations or lunar fortnights), tides may be predicted with any degree of accuracy. To do so, the following fiction is used. The actual tide curve, as obtained by plotting hourly observations (taken during the period named above), is assumed to be the resultant of a number of tide-waves superimposed, each having different *periods* between successive high waters, and different *ranges*, that is to say, differences of level between high and low water. The range of each *individual* tide is assumed to be constant. The *periods* or times elapsing between the successive high water of each component tide, are also assumed to be constant, and to bear some fixed proportion to the real mean motions of the real sun and moon, such as a solar day, a lunar day, half a solar and half a lunar day, a fortnight, a year, and so on.

The rise and fall of each tide is assumed to obey the law of a harmonic curve or oscillation, a term which will be explained later on.

It is capable of demonstration that the elevations at opposite sides of the earth produced by the attraction of the moon, would be produced to the same degree and to the same extent if the moon were cleft into two equal parts, placed equidistant from the earth and opposite to each other. The moon and ante-moon would transit successively at intervals of twelve lunar hours, producing tides at these intervals. So also for the sun. Then, to produce the effect of diurnal inequality a further moon and sun are imagined, passing the meridian at intervals of twenty-four hours, and so on.

A number of imaginary stars are assumed, each producing a high tide, immediately under it but *not* opposite it. The period of these stars, or their times of apparent revolution, are assumed to be multiples or submultiples of the real mean motions of the sun and moon. Imaginary stars are also assumed, to take account of perturbations produced by the variations of the *rate* of movement of the moon and sun, and of their distances from the earth,

as well as by the retardation of the tide-wave in flowing up an estuary. The problem is then—

(*a*) To find by observation the range of the tide produced by each imaginary star.

(*b*) To find positions of the heavens occupied by each of these imaginary stars at some one instant of time, such as midnight on January 1, with regard to the position of the mean sun. This known, their position at any other time may be predicted, and the time of high water at any place which each produces, may be ascertained.

Harmonic Curves or Oscillations referred to).—Before attempting to give an idea of the solution we shall briefly discuss the forms of single and compound harmonic curves. If hourly observations were made of the sea-level above some datum plane, and then plotted vertically from a base-line, the ordinates representing height, and the abscissæ time, a line sketched through the successive points would form an undulatory curve having the character of a compound harmonic curve. A simple harmonic curve may be constructed graphically as follows:—

Describe a circle, and divide its circumference into a number of equal parts, say sixteen. Through the centre of the circle draw a horizontal line, and along it lay off a number of equal parts, numbering them 0, 1, 2, 3, . . . 16. Mark also the division points of the generating circle 1, 2, 3, . . . 16. Then through the divisions of the horizontal line draw verticals, and intersect them by horizontal lines, drawn through the corresponding division points of the generating circle. Sketch a fair curve through the points of intersection. This curve will be a simple harmonic curve (*vide* Fig. 94).

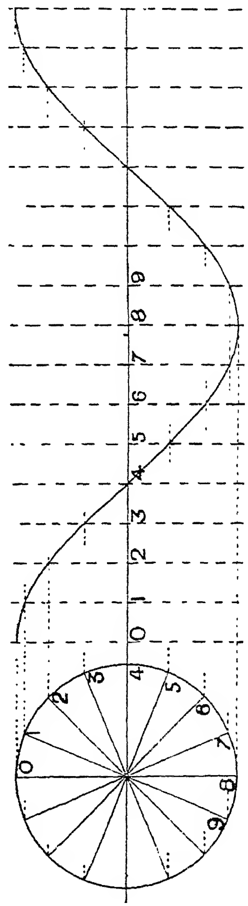


FIG. 94.

Application of Construction to Tidal Curves.—Now, let the diameter of the generating circle represent, to some suitable scale, the range of a tide, that is to say the difference of level between high and low water. Let the circumference of the generating circle represent (like a clock dial) the "period" of the tide, that is the time-interval between the instants of two successive high waters. For example, if it be desired to describe a lunar semi-diurnal tide-curve, the *whole* circumference will represent *half a lunar day* (12 h. 25 m. *solar* time). If it be divided into *twelve* parts, each will represent one lunar hour, about sixty-two solar minutes. With the same

scale used to lay down the range of the tide, lay off *downwards* the height of mean sea-level above the datum or zero plane. Draw a line through this point parallel to the line of mean sea-level, and produce the ordinates of the curve to cut it. The ordinates of the curve, or heights of intersecting points will represent the height of the water above datum, at successive *lunar* hours, counted from the instant of high water. If the horizontal time-intervals were each subdivided into sixty equal parts, then the heights of the water could be scaled off for intermediate *lunar* minutes, but what is usually required is the height at *mean solar* hours. To obtain this information it is merely necessary to regraduate the horizontal line, with equal and smaller intervals in the ratio of a lunar to a mean solar hour, *i.e.* in the ratio of 62 to 60 (approximately). Suppose, for example, that the horizontal scale of time is one inch to one hour of *mean solar time*, then, for laying off the lunar semi-diurnal tide-curve, the horizontal time intervals would be each $\frac{62}{60}$ of an inch. Having plotted the curve with abscissæ intervals of this length, the height of the water-level at any solar hour interval (counting from time of high water) could be scaled off by drawing ordinates at intervals of one inch, and so also for the intermediate minutes.

Compounding Harmonic Curves.—Suppose now that a lunar semi-diurnal tide-curve were traced in the manner described, not merely for a single lunar day, but repeated day after day for a considerable period, say twenty-eight days, the time of high water being shifted forward by about fifty minutes daily. Suppose that the horizontal hour intervals were each $\frac{62}{60}$ of an inch (or 1 inch = 1 *solar* hour), also that the scale for heights were 1 in. = 10 ft., then, to plot a semi-diurnal solar tide-curve (comparable with the lunar semi-diurnal tide-curve just described), the same *scale* must be used for the diameter of the generating circle, representing the range of the semi-diurnal solar tide, but the horizontal hour graduation would be one inch, thus representing *mean solar* hours. Inasmuch as the solar semi-diurnal high water will occur at the same hours, day after day, there will be no necessity for repeating this curve for a series of days, as in the case of the lunar tide curve. Having thus plotted the two curves to the same scales of time and height, they may be combined in the following manner.

Draw a horizontal line to represent mean sea-level, and divide it into equal parts, each representing one *mean solar* hour, twenty-four spaces representing an ordinary day. Number the points of division 1, 2, 3, 4, ... 12 noon, 1, 2, 3, 4, ... 12 midnight, or from 24 midnight to 24. Draw ordinates through the hour-points, *above* and *below* mean sea-level. Continue the graduations for a number of days, at least fourteen. Next, scale off the ordinates of the lunar semi-diurnal curve at each successive *solar* hour-division, measuring them, not from the datum line but from the line of mean sea-level. Now scale off the ordinates of the solar semi-diurnal curve at hour-intervals, measuring them also from the line of mean sea-level. When the hourly levels are both *above* or both *below* mean sea-level, add them together and plot the sum *above* or *below* the new line of mean sea-level at the appropriate hours. When one is *above* and one *below*, take the difference, and plot it *above* or *below* the mean sea-level line, according to the direction of the greater ordinate. The addition or deduction may be performed graphically by means of a pair of dividers.

Sketch in a fair curve through the points thus determined; a compound harmonic curve will be obtained, and the datum or zero line may then be drawn at its proper distance below mean sea-level. The resultant curve indicates the phenomenon of "neaps" and "springs," also that of "priming" and "lagging." The curves for successive days are assumed to be plotted vertically under one another in all cases.

It will be observed that the lines, traced through successive instants of high and low water, are not, as in the case of the semi-diurnal lunar and solar tides, straight lines, inclined in the former and vertical in the latter case.

Next, the student may draw a *diurnal* lunar tide, one having one high and one low water in the course of twenty-four lunar hours.

It has the same characteristics as the *semidiurnal* lunar tide, with the exception that there is but one high water and one low water in the course of twenty-four lunar hours.

If this be combined with the previous curve, the resultant curve exhibits clearly the diurnal inequality produced by superimposing a diurnal upon two semi-diurnal tides. The resultant curve resembles tide-curves that have been observed in many parts of the world.

The results of observation may be followed and duly represented by the addition and combination of other curves. One could combine a diurnal solar curve, a fortnightly lunar curve and a monthly lunar curve, representing the perturbations produced by the difference of the moon's distance from the earth, and so on.

By the assumption that any observed tidal oscillation may be resolved into a number of simple oscillations, having known periods, but epochs and amplitudes determined by observation, the most complex phenomena may be analysed and predicted.

Analysis of Tidal Observations.—No attempt will be made to describe the method of analysing tidal observations, that is to say, deducing from them the various components. The progress is not difficult but is extremely operose. The number of constants required usually amounts to 24, and for the Indian and other tables it is proposed to increase the number to 36. Another reason for omitting a detailed description of the analysis of observation is, that constants would be of little use to the surveyor, because the direct calculation of the predicted times and heights of high and low water is far too lengthy for him to undertake. Such predictions are in practice effected by a machine, an idea of which will be given later on.

The following considerations will, however, give a crude idea of the process of analysis.

The hourly reading of the gauge, either taken directly or scaled from the curve of a self-registering gauge, are tabulated in vertical columns. Each vertical column represents twenty-four mean solar hours, and is numbered from 0 at the top to 23 at the bottom. There are 369 columns, each representing a mean solar day. In each column the tide level is inscribed against its appropriate numbers. The columns are then summed vertically, and also horizontally. The sums of each day's readings appears at the bottom of the columns. The sum of all of the readings for 0 hours, 1 hour, etc., are inscribed at the right of the table. The sum of the sums of the vertical and horizontal summations must agree, if the arithmetic is correct.

The sum of all the sums, divided by the total number of observations, gives at once mean sea-level on the scale of the tide gauge.

Dividing the sum of each horizontal line by the number of observations in that line (369), then the average level at the solar hour represented by that line is obtained. Thus the mean of all observations taken at 3 p.m. on consecutive days gives the mean level of the sea at 3 p.m.

Now the moon and its derivatives have, during the 369 days, been in all possible positions with regard to the sun; so that their perturbing influences, sometimes adding, sometimes deducting, balance each other. The result is that the means of the twenty-four horizontal columns represent the ordinates of the primitive semi-diurnal solar tide, and at once give one component.

If the means for the hours 0 to 12 differ from those for the hours 12 to 24, this will also give a diurnal tide.

Suppose that, having got a continuous tidal curve, we lay off longer intervals representing lunar hours, and proceed in the same way.

We can thus obtain the averages for lunar hours and the elements for semi-diurnal and diurnal lunar tides, and so on for longer periods.

Similar constructions give other components.

The table on p. 260 gives the components for several places.

The Tide-predicting Machine.—Having obtained the "components," that is to say, the range of the various tides, generated by the imaginary stars, and their relative positions on a given day, the prediction is effected by means of the tide-predicting machine, the principle of which may thus be briefly described.

Cranks and slotted links for automatically tracing harmonic curves are provided, equal in number to the number of components to be used. The crank-axles of the several cranks are geared to a common shaft by means of toothed wheels, so proportioned that each crank revolves at the rate proportioned to the astronomical rate of revolution of the imaginary star which it represents, movement of the common driving-shaft being taken as unity. The lengths of the cranks are adjustable, according to the range of the component and they may be clamped in any desired angular position on their crank-shafts.

Each slotted link carries a pulley attached to the summit of the vertical spindle which determines its up-and-down motion.

A fine cord, fixed at one end, passes down under the first moving pulley, over a fixed pulley, under a second moving pulley, and so on, to the end of the whole series of moving pulleys, finally

passing over a fixed pulley, and carrying at its free end a weight with a pen or pencil, marking on a drum geared in proper ratio to the common driving-shaft.

Each moving pulley, as it ascends and descends owing to the revolution of its crank, describes a simple harmonic motion. The continuous cord combines the motions of all the pulleys, so that the resultant motion of the marking-pen traces a compound harmonic curve on the paper as it revolves.

The lengths of the several crank-arms are adjusted by means of a micrometer screw, so that each is proportional, on some common scale, to the range of the component tide. The angular positions of the several crank-arms are then adjusted so that each occupies the angular position of the component or phase of the imaginary stars, which it represents, at some given instant of time. The machine is then set in motion. The pencil traces on the drum a curve representing exactly the curve which would be traced by a self-recording gauge during any period. From this curve the times and heights of high and low water may be scaled off and tabulated. The curve for a whole year is traced in about four hours.

The annexed table (p. 260) shows some results.

Table of Some Results.—The principal components in general, taken in order of their effects, are: lunar and solar semidiurnal, longer elliptic semidiurnal, luni-solar diurnal, and lunar and solar diurnal (declinational); but, as will be evident from the table, this order does not hold universally.

The "epoch" of the component means the angle between the meridian of the place and the imaginary heavenly body which produces the component, at the instant when the latter is at its maximum.

Thus, dividing the epoch by the hourly motion, we get the time in hours between the time of transit of the imaginary heavenly body over the meridian of the place, and the instant when the tide due to that component is at its maximum at that place.

The period (in mean solar hours) of any one of the tidal components can be found by dividing 360 by the movement of the heavenly body per hour, given in the fourth column.

Horizontal Movement of Tides. Tidal Currents.—Hitherto the vertical movements or rise and fall of tides have alone been considered. It may not be out of place to make some observations as to the horizontal movements, or "tidal currents." The tide-wave differs little in its mechanical properties, except in magnitude, from an ordinary rolling wave, such as the ripples produced by throwing a stone into the water.

TABLE OF MEAN AMPLITUDES AND EPOCHS OF TIDAL CONSTANTS (FOR ADEN AND LIVERPOOL) FORMING THE SETTINGS OF THE TIDE PREDICTOR.

Period.	Description.	Conventional letter.	Movement of imaginary heavenly body in degrees per mean solar hour.	Aden. Amplitude or semi-range in feet.	Aden. Epoch or hour angle of maximum effect of the component.	Liverpool. Amplitude.	Liverpool. Epoch.
Sidereal day	Luni-solar diurnal . . .	K_1	15·041	1·30	35°	0·36	192°
½ (sidl. day)	Luni-solar semidiurnal .	K_2	30·082	0·20	240	0·94	1
½ (lunar day)	Lunar semidiurnal . .	M_2	28·984	1·57	226·5	9·98	320·6
¼ (lunar day)	First overtide of semi-diurnal	M_4	57·968	0·01	313	0·69	211
⅙ (lunar day)	Second ditto	M_6	86·952	0·01	345	0·20	331
	Smaller elliptic semi-diurnal	L_2	29·528	0·04	225	0·53	329
	Larger ditto	N_2	28·440	0·43	221	1·90	299
	Lunar diurnal (declinational)	O_1	13·943	0·66	37	0·37	38
	Solar ditto	P_1	14·959	0·39	31	0·13	182
Solar day	Solar diurnal) chiefly meteorological)	S_1	15·000	0·09	171	—	—
½ (solar day)	Solar semidiurnal . . .	S_2	30·000	0·69	246	3·16	6
	Smaller elliptic lunar diurnal	J_1	15·585	0·09	46	very small	
	Larger (ditto)	Q_1	13·399	0·15	38		
	Larger elliptic solar semi-diurnal	T_2	29·959	0·05	231	0·24	227
	Variational lunar semi-diurnal	U_2	27·968	0·08	190	0·26	34
	Larger evectional lunar semidiurnal	V_2	28·513	0·10	225	0·53	286
	Compound luni-solar ¼ diurnal	MS_4	62·032	0·01	155	0·41	258
	Compound luni-solar ½ diurnal	2 SM	31·016	0·02	101	not evaluated	
	Larger elliptic semidiurnal, second order	2 N	27·895	0·09	191		
	Compound elliptic lunar (¼ diurnal)	MN	57·424	0·04	17		
	Compound luni-solar declinational (⅓ diurnal)	M_2K_1	44·025	0·02	265		
	Ditto	$2M_2K_1$	42·927	0·01	9		
Solar year	Solar annual (chiefly meteorological)	Sa	0·041	0·37	355	0·40	250
½ (Solar year)	Solar semi-annual (chiefly meteorological)	Ssa	0·082	0·13	128	0·16	201

In a rolling wave, the particles in each horizontal parallel plane move with equal velocities in elliptical orbits. Proceeding downwards, the successive particles move in more and more flattened and shortened ellipses, till at the bottom the movement is in a horizontal line.

At the crest of the wave the forward motion of the particles is in the direction of the wave's motion, and is a maximum in that direction. At its lowest point, the motion is in the direction *opposite* to the wave and at its maximum also. The horizontal motion of the particles, at their mean position as regards altitude, is zero, but the vertical motion is at a maximum, falling when in front of the crest of the wave, rising when behind it in its line of motion.

Change of Tide Stream in Open Sea.—These considerations lead to the conclusion that in deep and open water the tide-stream would be in the direction of the advance of the tide wave, and will have a maximum velocity at high water, and would be also a maximum in the opposite direction at the instant of low water. Further, that there would be "slack-water," or no current, at the instants of "half tides," that is to say, when the water-level at the point of observation coincides with mean sea-level. In short, slack water in the open sea occurs at half tide.

Tide Motion in Bays and Lagoons.—These simple motions are, however, profoundly complicated by the form of the coast, in bays, estuaries, and rivers. The following considerations, however, serve to give an idea of the phenomena which may be expected to take place under several conditions.

Let AB (Fig. 95) be a straight coast-line plunging abruptly into deep water. Let the direction of the advance of the tide-wave

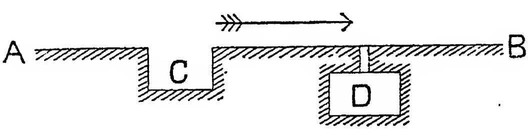


FIG. 95.

be that of the arrow. At C assume a small inlet of no great length inland from the coast-line, wide at its mouth compared with its length inland, like a tidal dock, and being deep throughout in comparison to the rise and fall of the tide outside. Then it is clear that at the inland extremity there will be no current, the water-level rising and falling to the same extent practically as the tide in the open sea. At the mouth there will be a slight in and out

current, *in* during the "flood" or "rising tide," *out* during the "ebb" or "falling tide."

In the open, just outside the mouth of the inlet, the in-and-out going current will be deflected in the direction of the tidal current outside—the direction of the tide wave from half flood to half ebb; and against it from half ebb to half flood again.

Next consider a pond or dock connected to the open sea by means of a very narrow channel, such as a pipe or culvert, of so small a section as to require an appreciable head of water to establish a strong current through it, so much so that during the rise of the tide the level of the water in the basin never rises up to or falls down to the level of the sea at high or low water outside.

It is clear that the instants of high and low water will be later inside the basin than outside. By the time there is high water outside the water inside will only have risen to some lower level. The inflow will continue for some time after high water has occurred outside, and until the level of the water outside has fallen to the level of the still rising water inside, when the current through the channel of communication will stop and reverse its direction. A similar effect will be observed on the falling tide. This is a pure case of "in-pouring." The change of direction of the current occurs at high and low water *inside*. An effect of similar character is actually observed in the case of lagoons connected by narrow outlets to the open sea. In such cases the phenomena are complicated by the fact that the waterway and consequent discharge vary with the height of the tide.

Tide Motion on a Shelving Beach.—The case of a shelving foreshore may next be considered. Let Fig. 96 represent a section

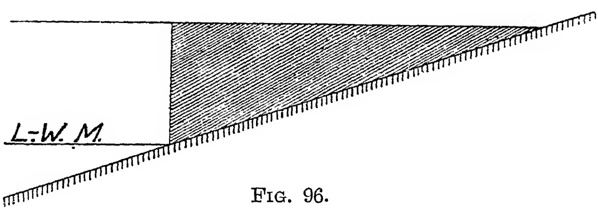


Fig. 96.

of such a foreshore. The shaded portion has to be filled and emptied during the rise and fall of the tide. This will occasion an indraught during the rise and an outdraught during the fall of the tide. These motions will, however, be compounded with the motions of the tide-stream in the offing.

The following motions may therefore be expected. Let Fig. 97 represent the plan of the coast; the feathered arrow, as before, indicates the direction of the tide-wave in the offing.

At low water the tidal current will be in the opposite direction to the motion of the tidal wave, and as the water-level is for a short time stationary, there will be no indraught. At half flood there is slack water in the offing, but the tide is still rising, with its maximum vertical motion. There will therefore be no current parallel to the shore, but simply an indraught perpendicular to the shore-line. As the tide continues to rise, that is after half flood,

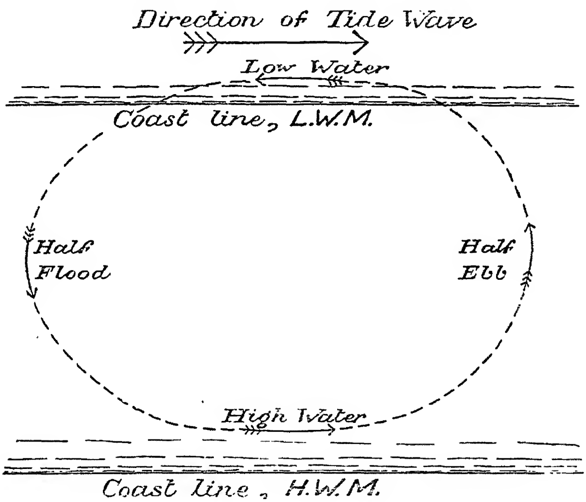


Fig. 97.

the current parallel to the shore accelerates. The inpouring retards, until at high water the current will be parallel to the shore and in the direction of the advancing tide-wave. As the tide falls the outpouring motion again sets in, until at half ebb the current will be simply outward from the shore.

A floating body on a uniformly shelving foreshore may therefore be expected to describe an elliptical course.

In nature, however, uniformly sloping and straight foreshores rarely occur.

In practice the only safe plan is to determine the currents by actual observation. The abstract considerations put forward in the preceding sections only serve to indicate what is likely to

happen, and the general character of the phenomena to be looked for.

In the case of an estuary, the action of the tide-wave and of the currents which it induces, is a resultant of the several actions which have been described. These are further complicated when a river of considerable volume enters the head of the estuary.

Estuary and Fluviatile Section defined.—A river may be divided into an estuary section and a fluviatile section. The estuary section may be defined as the part nearest to the open sea, where—

(*a*) The bottom is below mean sea-level, so that there is a considerable sectional area at low water.

(*b*) The flow of the river is negligible in volume compared with the volume of tide water which enters and leaves at each successive tide.

The fluviatile section is that in which the bottom approaches to or rises above low-water mark, and in which the flow of the river bears an appreciable relation to the volume of the entering and outgoing tide water.

Tidal Motion in the Estuary Section.—In the estuary section the inpouring action predominates. The inpouring effect on the rise of the tide produces an inward current. To generate the current, that is to say to overcome the inertia of the particles of water, and further to overcome the frictional resistance of the sides, there must be a surface slope from the mouth inland. As the tide rises the acceleration becomes greater, and a wave-like action is set up which often rolls onward into the fluviatile section. As the estuary becomes narrower the kinetic energy of the mass of water increases. The wave action becomes more and more prevalent till at a certain distance up the estuary the level of high water may be actually higher than that of the corresponding tide at the mouth of the estuary. A similar action takes place on the falling tide (or ebb). The results of these actions are—

(*a*) That the instants of high and low water are respectively later as the estuary is ascended than the corresponding times outside.

(*b*) That in a free and uniformly contracting estuary, the tide-range at some point near its head may be actually greater than it is in the sea at the mouth, high-water mark being higher and low-water mark lower at the upper station than at the lower for any corresponding tide.

(*c*) Proceeding up the estuary, beyond some critical point of maximum range, frictional resistance diminishes the range of the tide, the kinetic energy of the water particles expending itself in friction and eddy motion.

For example, the range of spring tides at London Bridge is about 22 feet, whilst at Sheerness it is only 15 feet, and at Ramsgate somewhat less again.

(*d*) In the estuary, owing to the predominance of the inpouring action, the direction of the current changes at or near the instant of high or low water, respectively, at any given point of observation. Nevertheless, it is often observed that the flood stream continues in an upstream direction for some time after the instant of high water. In the Thames, watermen often say that "the tide has made its mark on the shore, but is still rising in mid-stream." The fact being that the flood or upstream current, as indicated by the swinging of vessels at anchor, and the motion of floating objects, continues owing to momentum, for an appreciable period of time after the water-level has fallen, as indicated by the line of detritus which is left by the falling tide on the beach or "hard." This phenomenon is less noticeable on the ebb.

Tidal Motion in the Fluviatile Section.—The wave generated by the inpouring action in the estuary section rolls on into the fluviatile section, diminishing in range, partly on account of fluid friction, and partly from rise in the bed of the river. The distance to which it will penetrate into the river depends upon the area of cross-section and upon the freedom from frictional resistance, owing to bends and other obstructions. The period of rise becomes shorter than the fall. The whole period of rise may occupy three hours or less, the fall nine hours or more. Towards the latter part of the tidal period the level is nearly constant, coinciding approximately with the flow of the river, or what it would have been had it been free from tidal influence.

Mean Sea-level. Tidal Observations to be referred to a Datum Plane, such as Mean Sea-level.—The main object of the preceding discussion is to emphasise the necessity for referring the levels of the bed and banks of estuaries and rivers to some one fixed datum plane. "High-water mark" and low-water mark are not fixed planes, even for one place. The range of successive tides, and consequently the levels of successive high- and low-water marks, varies from day to day, as has been already shown. Mean sea-level, on the other hand, in the open sea, as the average of continued hourly observations extending over 369 days, is a fixed level all over the earth; it is neither more nor less than the geodetic surface of the earth, as defined in the chapter on this subject (Chap. VI.).

The Disadvantages of referring to Low-water Mark as a Datum.—If tide-gauges be observed at sundry stations up an estuary, it may often be found that up to a certain point the

mean water-level would coincide with mean sea-level at the open seas, though high water might be higher and low water lower than at the mouth. The limit of coincidence of mean water-level with mean sea-level, as determined by accurate levelling, might be used as a definition of the limit of the estuary section. Beyond this point the fluviatile section commences. Here mean river-level no longer coincides with mean sea-level. If the level of the bed were expressed as depths below a low-water mark, as is done for obvious and sufficient reasons in navigation charts, then an erroneous idea of the levels would be formed. The only safe plan for engineering purposes is to refer *levels* of the river-beds to some fixed plane, such as mean sea-level, or to a plane at a fixed distance above or below it. This done, if a navigation chart is to be prepared a more or less arbitrary plane of low water may be established for successive reaches of the river, and levels reduced to "soundings" or depths below these planes for purposes of navigation.

If, however, the depths were referred to low- or high-water mark without recording actual levels, erroneous conclusions as to the effect of engineering works might result. For example, take the case of a small and tortuous stream, choked perhaps with weeds. In the natural state of the river the tide would hardly be perceptible in the upper regions, the energy of both the tide-stream and the tide-wave being absorbed by various resistances. Now let the stream be cleared, its course shortened and improved by cutting off bends. The effect of these improvements would be to admit the tide-wave more freely than before. In the upper reaches the tide would rise higher and ebb lower than before. Vessels of larger draught could ascend the river at high water. If, however, the only record were depths at low water, one might infer that as the low-water depth in the upper reaches had become lessened, the bottom was rising by accretion, whereas perhaps the opposite effect was taking place.

Effect of Wind on the Tides.—The effect of winds upon the rise and fall of the tide is a subject on which there is great difference of opinion. The usual belief is that wind has an important influence on tides. Personally, the writer holds the opinion that wind has less influence than is generally supposed. He bases his opinion, firstly, on observed facts. The tides at several ports (Aden, Bombay, Singapore and Hong Kong) where he and others have made observations, present apparent anomalies, one of which is a large diurnal inequality (much larger than is usual in English waters).

The observations do not agree with the rules for tidal prediction usually given in works on navigation. These perturbations were

tacitly ascribed by mariners to be due to wind. In several places the variations were ascribed to the change of the monsoon. There were tides for the "north-east" and "south-west" monsoons respectively. Proper continuous observations having been made, however, and subjected to harmonic analysis, coefficients or factors were determined by which the time and height of high water are now predicted with accuracy, without reference to the effect of winds, though these varied greatly in force and directions at different seasons and from day to day. Mr. E. Roberts has informed the writer that he has often found that apparently abnormal tides, popularly ascribed to wind, were predictable when a suitable number of coefficients were used in the proper manner.

It is a matter of common knowledge that violent winds often prevail at the equinoxes (equinoctial gales). It seems, therefore, that the configurations of the moon and sun, which produce equinoctial spring tides, also produce atmospheric disturbances. The extraordinary high tides at such times therefore may be traced direct to the heavenly bodies and not to wind. The writer is fully aware of the fact, however, that extraordinary and unpredictable tidal perturbations accompany violent winds.

Barometric Effects.—The writer believes that the elevation of the water which in certain cases doubtless takes place is due to the great depression of the barometer, which actually occurs in the centre of a cyclone as it passes onward. The depression, in one instance within the writer's knowledge, amounted to 2 inches of mercury. That is to say, the barometer was 2 inches lower in the centre of the cyclone than at its margin. Now 2 inches of mercury is roughly equivalent to 2 feet of water; so that at the centre of the cyclone, supposing it stationary, the sea-level would be about 2 feet higher than at its circumference. The reduced pressure at the centre of the cyclone as it sweeps along may reasonably be expected to produce a large wave-like elevation following the course of the cyclone. If the great ocean tide-wave is only 4 feet, and yet suffices to generate tides ranging 20 feet and upwards, it is clear that a relatively small barometric wave may produce great variations in the ranges of tides in estuaries.

To conclude, the writer ventures to believe that wind in itself has little effect on the tide, but that large barometric depressions which usually accompany violent winds, may on the other hand have an important effect, though such effects are relatively of rare occurrence.

On Tide-gauges. The Float and Pole Gauge.—Fig. 98 shows a simple form of tide-gauge which has been found useful. It is suited for attachment to a pier or jetty extending into deep water,

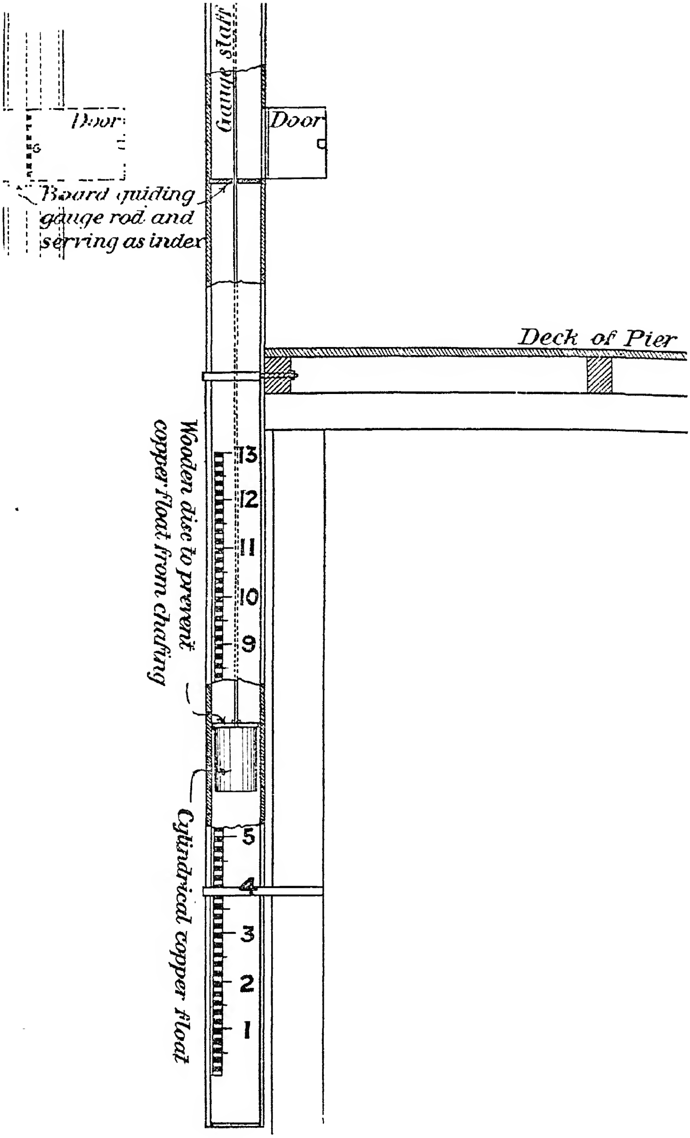

Gauge staff
Door
Door
Board guiding gauge rod and serving as index
Deck of Pier
Wooden disc to prevent copper float from chafing
Cylindrical copper float
13
12
11
10
9
5
4
3
2
1

FIG. 98

at low-water spring tides. The rod carried by the float is graduated from above downwards, and read by means of a fixed index, rendered visible by an opening or door in the side of the case. The case contains also a clock and a lamp for reading at night. To obtain records with such a gauge a man must be in attendance night and day.

The zero of the outside scales should be referred by levelling to one or more permanent bench-marks on shore. Then by reading the outer scale in calm weather the index of the float staff can be set so that the readings coincide.

The float staff is graduated from above downwards. This is, on the whole, the most convenient plan, because the reading always takes place at the same level. It has the disadvantage that inversions in reading the smaller subdivisions may take place, for example 9·3 might be recorded as 8·7. This has not in the writer's experience proved to be an important defect, for these errors may easily be detected and corrected. It might, however, be well to number each tenth as though it were a unit. For 9·3 93 would be inscribed.

The Self-registering Tide-gauge.—The self-registering tide-gauge is exceedingly simple in principle, though its use is not so elementary in practice.

A float, of large diameter in comparison with its depth, swims in a cylinder or well having a free communication with the sea, placed well below the lowest low-water mark. The opening must have sufficient area to ensure that the water-level in the well is, at all states of tides, identical with the sea-level outside. At the same time the opening should be small enough to prevent ordinary wind-waves from creating oscillations of short periods in the float-well. A cord, attached to the float, passes over or is coiled round a wheel, being maintained in constant tension by means of a counter-weight. On the spindle of the wheel a smaller wheel is keyed. Round the latter a cord winds which moves a sliding pencil-carrier. This cord is kept in tension by a counter-weight.

The pencil traces a curve upon a sheet of paper clamped on to a horizontal drum, which may be 2 to 3 feet long, and which is caused to revolve about its axis *once in twenty-four solar hours by means of a good clock*. The pencil thus traces, to a reduced scale, the tide-curve. Usually this paper remains on for a week or a fortnight, and the result is a series of interlacing curves.

Tide-gauge (Sir Wm. Thomson's Pattern).—The instrument consists of an astronomical clock, float-wheel and gear work for reducing the scale, and three drums, the whole fitted on a suitable plate and supporting standards, and requiring no further fixing.

The clock is fitted with a compensated pendulum, and serves to show the time and to drive the centre or main drum of the instrument. The float-wheel is provided with pins which guide the copper band of the float as it coils itself during the rising tide. The right-hand drum receives a reel of paper, and the paper is fitted to the instrument without further fixing. The left-hand or haul-off drum receives the paper records after it has passed round the main drum. The paper may be left to accumulate almost without limit on the haul-off drum, or can be removed at any time. The datum line on the record paper is traced by a fixed pencil, which can be adjusted to any level.

The main drum is furnished with a series of pins in distinctive positions, which perforate the paper as it passes over the drum, and thus give an absolute record of the time. The pencil-carrier is made to counterbalance the float-band when the scale is not too greatly reduced, in which case the weight of the float-band is partially relieved by a counterpoise weight acting on the axis of the float-wheel. The system of making the recording pencil balance the float-band is a great advantage, and accuracy of recording is secured.

The employment of a continuous roll of paper obviates the necessity of applying fresh paper to the recording drum, and the tide-gauge can thus be left untended, except for the purpose of winding the clock, for an indefinite period.

The usual height of the drum is 18 inches, and therefore with a range of 16 feet the scale is 1 inch per 1 foot.

The paper passes over the drum at the rate of 1 inch per hour, and a roll of paper lasts about 1 year. For further particulars see the *Proceedings of the Institution of Civil Engineers*, vol. lxv.

The cost of self-recording tide-gauges is great, and on account of the clockwork they require a firm foundation. They take more looking after than would be supposed. There is always a chance of the clock stopping, or of the string to which the float is attached breaking, thus interrupting the record and perhaps invalidating a whole day's work. On the whole the plain float and staff, read hourly or half-hourly by a suitable man, is the most practical plan, unless the operations are so extensive as to justify the erection of a permanent self-recording gauge, and the employment of a competent person to look after it. Even if such a gauge be provided it would be well to have a simple float and staff in addition as a stand-by in case of accident.

Though a year's observations will determine the constants fairly accurately, still it is satisfactory to have a second year of records as a check, and for the readjustment of constants. Three years'

observation at the outside will give all the information for the practical prediction of the tide and for the determination of mean sea-level. After this time further observations are merely of academic interest, such as the determination of tides of very long periods, the effect of great atmospheric disturbances, or the secular rise or fall of the coast. It does not seem, therefore, desirable to procure a costly instrument when it only need be used for a relatively short time.

Tide-gauges on Open Coasts.—It may be desired to take tide observations on an open beach, shelving gently seaward. To construct a jetty with a substantial head planted below low-water spring tides for the reception of the tide-gauge would probably be impracticable on account of cost, or for the same reason to

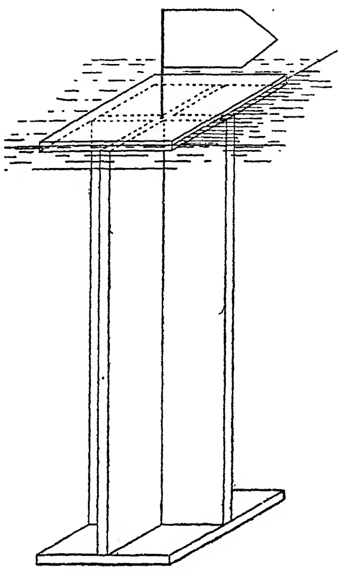


Fig. 99.

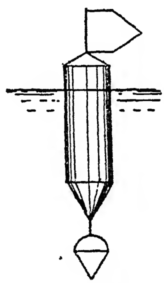


Fig. 100.

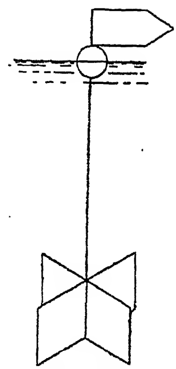


Fig. 101.

construct a low-level channel from a well on shore would also be impracticable. In such cases a syphon may be employed. The summit of the syphon pipe should be a foot or so below high-water mark neap tides, and a tap should be provided at the highest point to discharge air, and some arrangement should be made for scouring out silt which might accumulate in the pipe. (*Vide* "A Manual of Tidal Observations," by Major Baird, R.E., London, Taylor and Francis, 1886.)

A rod and float tide-gauge, similar to that already described, should be attached to a substantial pile driven about low-water

neaps, so that a check reading can be obtained at any time except that of extreme low waters. If the levels as recorded in the open water do not coincide with the levels in the float cylinder, it is a

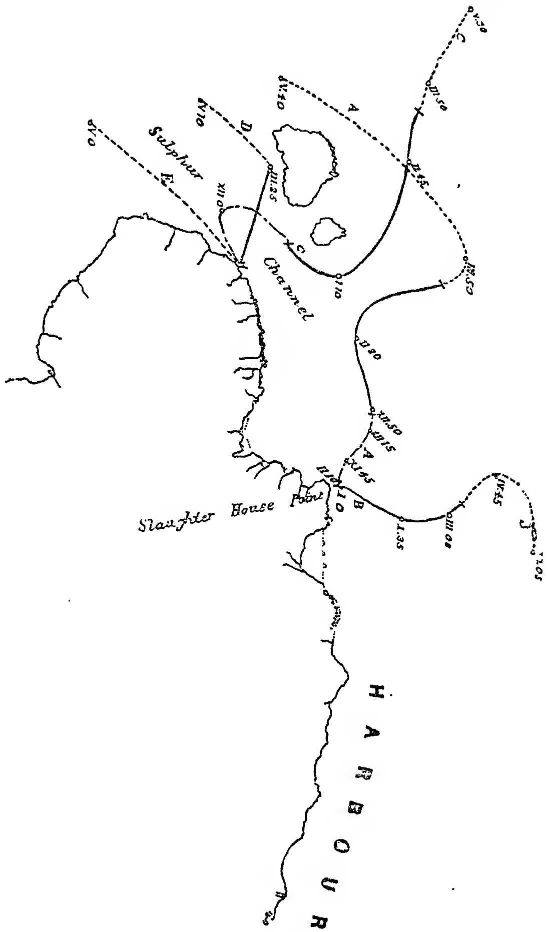


FIG. 102.

sign that the pipe requires cleaning, or that air has accumulated in the summit of the syphon.

Determination of Direction and Velocity of Tidal Currents.—It

is often necessary to determine the direction and velocity of tidal currents; such a case often occurs in connection with projects of sewerage. It is necessary to trace the path of the sewage at different states of tide when discharged from a proposed point of outfall. For this purpose a number of floats must be prepared. Figs. 99 and 100 show forms which have been found convenient. These floats are put in at the point to be investigated in succession at intervals of about an hour, and allowed to float away freely. They are then followed by a steam launch and their positions

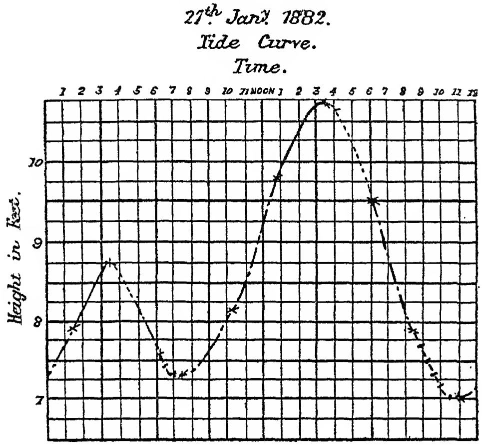


FIG. 103.

determined by sextant angles to known objects on shore, or these positions may be taken up by observations from two theodolites or two plane-tables on land. Each position is plotted forthwith, and the time of determination is noted against the point on the plan representing the corresponding position of the float. Through the points representing the successive positions of each individual float a fair curve is drawn to represent its path. This is done with each of the floats.

Meanwhile, a tide-gauge is read continuously at some base-station, and the hourly readings plotted as a curve. The path of each float is then determined and graduated into hourly spaces. The points so obtained represent the position of the float at equal intervals of one hour, and show the direction of its motion at the corresponding hourly heights of the tide. For recording the direction of the deep-water flow the float shown in Fig. 101 is useful.

Fig. 102 shows an example of this procedure. It is a copy of one day's observation taken at Hong Kong for the purpose of determining the suitability of a certain point of sewage outfall. In Fig. 103 the tide-curve for the corresponding day is shown.

These observations are to be repeated frequently at different periods of the lunation, both at neaps and springs, so that the currents may be determined at all states of the tide.

Combining a number of such observations, a general diagram could be prepared showing the direction of the currents on the flood and ebb respectively.

CHAPTER VIII

SURVEYS OF COAST-LINES, HARBOURS, ROADSTEADS, RIVERS, ETC.

Object of Chapter.—The object of the present chapter is to explain the special methods, artifices, and appliances used in preparing engineering plans of harbours, rivers, estuaries, and roadsteads.

It is frequently necessary to delineate the foreshore of coast-lines, harbours, and estuaries as defined by high- or low-water mark, as well as to survey in the spot soundings or submarine contour lines (as determined by soundings of equal depth). An accurate knowledge of the bottom formation is very necessary before designs can be prepared for works such as sewer outlets, wharf or bridge piers, dock wall foundations, etc. The method of procedure when preparing such plans will now be discussed.

Datum Plane, Soundings, and Levels.—In such plans, whether of ground above or below water, the levels given must be referred to some one datum plane, preferably at a height below the level of the bottom in the deepest water that occurs in the plan, and certainly below the level of the lowest constructions under consideration, such as the foundations of dock walls, wharves, etc.

This avoids the necessity for distinguishing between levels *above* and *below* datum. For example, in the survey of the foreshore of Bombay harbour the levels both above and below low-water mark were referred to what is known as General Delisle's datum, which is a plane 100 feet below a mark on a step of the town hall. Mean sea-level stands at 83·30 feet above this datum. Extreme low-water mark is about 72·00 feet and extreme high-water mark about 91·00 feet above this datum plane.

All the foundations of dock walls, even of the bottom of the harbour in the fair-way, have therefore in this case positive levels.

In this respect an engineering plan differs materially from a nautical chart, the object of which is to show the least depth of water that is likely to be found. In charts, soundings or depths are shown, and these are referred to *extreme* low-water mark.

Marine surveyors determine a low-water level for each chart.

Usually this is fixed for reference by stating that it is so much *above* or *below* a certain permanent mark, such as a bolt in a sea wall or on the sill of some dock, etc. By ascertaining the level of the mark, soundings may be converted into levels by deducting them from the level representing low-water mark.

Tide-Gauge.—The first step towards making a marine or river survey is to establish a tide-gauge or river-gauge at or near the scene of operations. These have been described in the last chapter. If a pier or wharf be available, then the gauge can be fixed at a place where there are always a few feet of water.

Having decided on the datum plane of the survey, such being at a certain number of feet below the level of a bench-mark on shore, the next step is to determine the level of the "zero" of the tide-gauge scale. This is conveniently done by driving a strong staple like the hinge of a gate (Fig. 104) into some solid and permanent structure not far from the tide-gauge, at or about mean sea-level so that it may be uncovered twice daily.

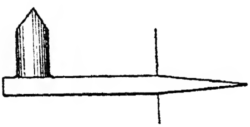


FIG. 104.

The level of the top of the pin is taken in the ordinary way. Then on a calm day the gauge is read at the moment at which the surface of the water coincides with the top of the pin. The reading of the gauge *deducted* from the reduced level of the pin gives the reduced level of the "zero" of the gauge. The reduced level of the water surface at any moment may be obtained by *adding* the gauge-reading to the zero level. The staple also serves as a point of reference whereby the "zero" of the gauge may be checked from time to time, to see whether it has shifted on account of the float having become water-logged, or from any other cause.

If there be no solid structure, such as a pier or wharf extending below low-water mark to which a tide-gauge can be attached, then a series of gauges may be used. Stout stakes may be driven firmly home into the ground, to which graduated scales are to be attached, as shown in Fig. 105. These scales must be carefully adjusted with a level and staff as the tide recedes, so that their divisions coincide with reduced levels, and so that the bottom division of each stake coincides with the top division of the next lower, and so on.

If there be much boat traffic it will be prudent to make the stakes numerous and short, for if long they would be liable to be knocked down, or might cause damage to craft.

Survey of Foreshore.—Having established the tide-gauge, a survey is made of the foreshore or coast-line, or of the banks of a

river, by any of the ordinary methods of surveying. Such detail as may be desirable will be delineated, such as sea walls, wharves, detached buildings, and the like.

As far as the marine survey is concerned, it is merely necessary to fix numerous points along the foreshore. If the survey is to extend out to a considerable distance seaward, it is desirable to fix by intersections the position of conspicuous objects such as steeples, chimneys, flagstaffs, lighthouses, conspicuous trees, and the like. Prominent rocks may be cut in and made conspicuous by a coating of whitewash, or piles of stones may be erected and whitewashed.

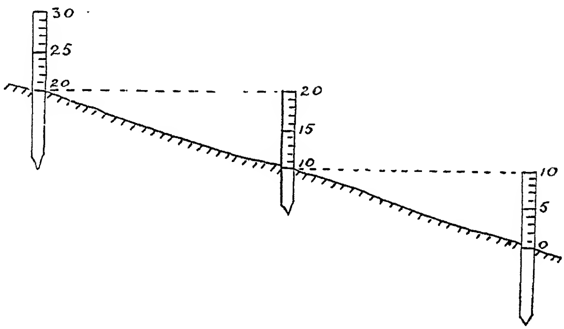


FIG. 105.

Sections of Foreshore.—The next step will be to take numerous sections of the foreshore down to low-water mark. This is done with the ordinary level and staff. The section lines may be set out with a theodolite or pocket-sextant, making some definite angle with the terrestrial survey lines, and connected to the survey points by chain measurements.

The distances from the survey line to the successive positions of the staff may be measured with the ordinary chain or steel band, or they may be determined tacheometrically or by subtense measurements, or by triangulation.

Levelling Operations.—Levelling operations should be extended down to low-water mark, and even below, to the extent that the staff-holder and surveyor can wade in.

Levelling is more accurate than sounding, and more easily reduced and plotted. Advantage should be taken of low tides to level as far as possible.

It often happens that very low tides occur at night. Levelling may be easily conducted at night by providing a means of illuminating the field of the telescope, so as to render the cross-hairs

visible. This may be done by soldering to the dew-cap of the telescope a mirror of silvered brass or clean tin-plate, making an angle of 45° with the axis of the telescope. An elliptical hole is cut in this mirror, the shorter axis of the ellipse being about three-fourths of the diameter of the object glass. The annular mirror reflects the light of a lantern or torch and renders the wires visible, whilst the staff, illuminated by a torch, is visible through the hole in the mirror. A slip of white drawing-paper about $\frac{3}{16}$ inch wide, tied to the dew-cap and bent 45° across the object glass, answers very well.

Sounding Operations.—Sounding is conducted on similar principles to levelling. The surface of the water corresponds to the line of sight of the level. The water-level is determined by the tide-gauge, and by deducting from its readings the synchronous depths of water the spot levels of the bottom are obtained.

An observer is stationed at the tide-gauge, and records its readings half hourly, or more frequently near high and low water. The time of taking each sounding is noted also. The water-level at the moment of sounding can be thus ascertained from the final tide-gauge record by interpolation. If the tidal range is great, the tide-gauge readings may be plotted as a curve and the level scaled off at any required time.

In the case of open coasts, bays, or harbours, a single tide-gauge will serve for an area of several square miles. The level will be practically the same at all points at the same instant of time. Not so, however, in the case of rivers, creeks, or estuaries, where the water-level at the same instant will vary very materially at points not far apart. In such cases, therefore, it is necessary to have numerous subsidiary tide-gauges, and to use the readings of each one separately for the reduction of the soundings taken in its immediate vicinity.

Staff for Soundings.—For depths not exceeding 15 feet, a graduated staff is preferable to a sounding line. It is well to weight the bottom with lead to such an extent as to cause the rod to float upright. If the bottom be soft, the rod should have a flat foot. Longer rods may, however, be used if the lines of soundings be taken, as hereafter described, parallel to the stream line, the boat being allowed to drift with the stream.

Steel Bands or Wire as Lead-lines.—For greater depths, a steel band or wire with a 7-lb. weight attached to it may be used, or a common linen tape will serve if its length be checked. In deep water, and especially in a strong run of tide, a steel piano-forte-wire graduated by attaching small brass labels to it can be used with advantage, as it offers less resistance to the water than

a flat tape, and so can be more easily brought into a perpendicular position. An ordinary 100-ft. chain with a 7-lb. weight at one end makes a fairly efficient sounding line.

The ordinary hempen lead-line stretches so much and requires such frequent correcting that it is unsuitable for accurate work. In the offing, where an error of a foot is unimportant, it may be conveniently employed. It should be graduated to single feet, and not to fathoms only.

Field Book.—The field book for recording soundings should have the following principal headings :—

Distances.	Time. h. m.	Soundings.	Tide-gauge.	Reduced levels.	Remarks.

The tide-gauge readings are filled in on the completion of the day's work.

The object to be aimed at is to cover the bottom of the submerged area with spot levels. In order to ensure regularity, the soundings, are usually taken at equal intervals of from 50 to 100 feet in straight lines parallel to each other. Where rocks are suspected to exist, the intervals should be reduced to prevent a danger from escaping detection.

Fixing the Positions of Soundings.—To fix the position of the soundings, the first step is to put in pegs along the survey lines on

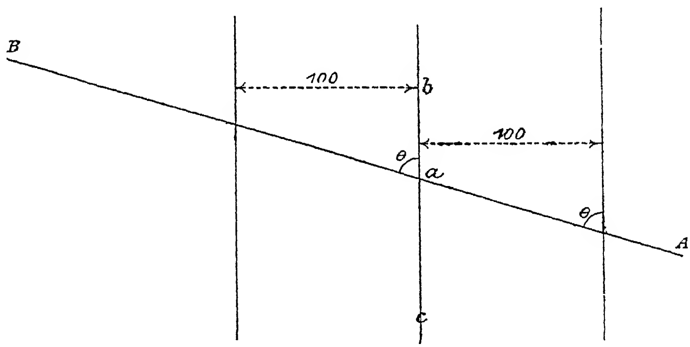


FIG. 106.

shore at the desired distance apart. Generally, it will be well that the lines of soundings on an open coast should have some definite bearing, say north and south, or east and west, or some bearing at

right angles to the general trend of the coast. The lines of soundings should be taken at some fixed distance apart, such as 100 feet. The bearings of the survey lines being known, it is only necessary to set out along the survey line successive distances of 100 cosec θ, where θ is the angle between the desired bearing of the line of soundings and that of the survey line (*vide* Fig. 106). Then, with a theodolite or sextant at a, points b and c can be fixed, so that the line bc makes the desired angle with the survey line AB. Flags or signals can then be put up at b and c. By keeping these signals "in one" the boat may be steered on the desired line of soundings.

If a river or estuary is to be surveyed, it is desirable that the lines of soundings should be at right angles to the central line of

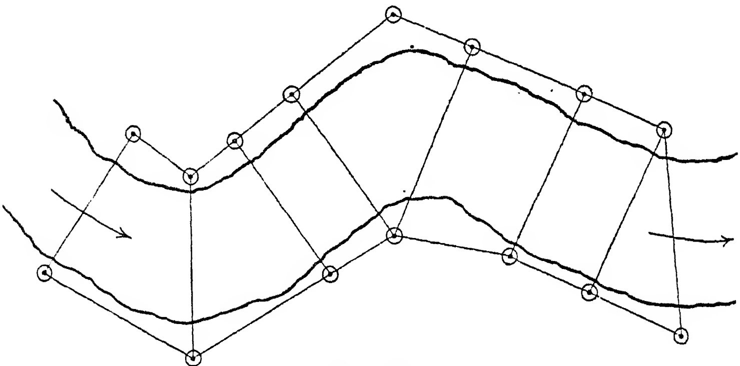


Fig. 107.

the stream. These lines may be set out at right angles to the survey lines (Fig. 107), determining the shape of the bank and checked by measurement along the survey lines on the opposite bank. Or, if the terrestrial survey has been plotted, giving a general outline of the river-bed, the positions of the lines of soundings may be selected on the plan, and the positions of their intersections with the survey lines can be scaled off, and set out on the ground.

Determination of Distances of Soundings from the Survey Lines.—To determine the distances of the several soundings from the survey lines, a thin steel wire rope about $\frac{1}{8}$ inch or $\frac{3}{16}$ inch diameter is often useful. This rope is well stretched and graduated to intervals of 50 to 100 feet, either by attaching small brass labels or by wrapping round it a small coil of brass wire and securing it with a drop of solder. The knots thus formed may be distinguished by pieces of string, with knots tied to indicate the

number of feet. The steel wire rope is strong, and therefore can be strained practically quite straight. A length of a thousand feet may be easily manipulated, so that where there is not much traffic the rope may be stretched from bank to bank. In other cases, the outer extremity may be attached to a boat moored in the desired line. The wire rope having been stretched out, it is under-run with a boat and a sounding taken at each knot. The intermediate distances may be measured with a tape or 10-foot rod.

If there be much traffic it will be impossible to use a rope at right angles to the stream, and also if there be a strong run of tide the manipulation of the rope is laborious, and much time is lost in running it out, straining it, and mooring and unmooring the boat. So much so, that sounding can in such cases only take place at slack water. As sounding can only be conducted properly when the water is smooth, it is desirable that some method should be adopted which will enable it to go on at all states of the tide.

Measurements in a Stream or Tide-way.—In a stream or strong tide-way, the wire rope may be used conveniently by running it in the direction of the current. In this position also it is less liable to be fouled by passing vessels. The following method has been found convenient in the case of a harbour survey, A piece of log line 2000 feet in length was prepared, and marked at intervals of 100 feet by knotted strings. This line stretched so much that it could not be used as a measure. It was stretched, however, between two anchored boats, A and B, so that the length of the line was in the direction of the current. The position of each boat was fixed by observations to points on shore, and soundings taken along the line at each knot.

The soundings were plotted from the determined positions of A and B, the distance being divided into a number of equal parts, corresponding with the number of soundings. It was rarely found practicable to keep the line quite straight, owing to cross-currents, or to the steering of the sounding-boat, and therefore at least one intermediate position was fixed by observation, and the soundings were plotted along a fair curve sketched through the fixed points, spacing them by dividing the line into as many parts as there were knots between the boats.

A boat or a buoy was fixed at C for a new line CD, and the next line was taken in a similar manner.

The position of soundings may be fixed by means of two theodolites or two plane-tables set up over known points on shore. Or if the soundings are taken along a definite line indicated by signals or marks ashore, one theodolite or one plane-table would suffice (*vide* Fig. 108).

When the boat is brought so that the sounding-rod or line is in line with the two flags or marks on shore, the man in the boat gives a signal and the pole or a small flag in the boat is intersected by the cross-wires of the theodolite and a bearing is read off, and ultimately plotted by protractor. With the plane-table, a ray is drawn intersecting the prolongation of the line joining the two flags on the sheet.

The accuracy of this method depends upon the distance between the two flags and the magnitude of the angle at the vertex, *i.e.* the boat. If the two flags are near together, the

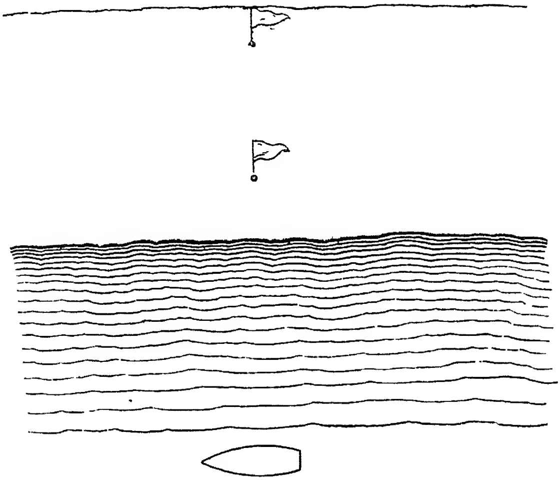


FIG. 108.

direction of the line of soundings will be imperfectly defined. If the angle at the boat be too acute, the intersection of the rays will be uncertain.

The co-operation of a second observer on shore is inconvenient, and may be dispensed with by the use of the sextant in the boat. The surveyor is then independent. It is evident that if at the moment of sounding the signals A and B (Fig. 109) be exactly in line, and the angle AXC is observed at the boat, the position of the latter is fixed. A second angle AXD would give a check. Indeed, by observing two angles such as BXC and BXD the position of the boat at X is fixed by the three-point method, independently of the prolongation of any fixed line such as AB. It

often happens that it is impossible to lay out a line AB of sufficient length to be of real use. Setting out the line on shore takes time, so that, on the whole, the three-point method is the most generally useful. Having determined the exact position of several points on shore, signals are put up at once to permanently mark them. The signals must be conspicuous, far more so than for observing with a theodolite, and care must be taken to suit the colour to the background. Vision is not so distinct with the sextant as with the theodolite. In the above manner of fixing

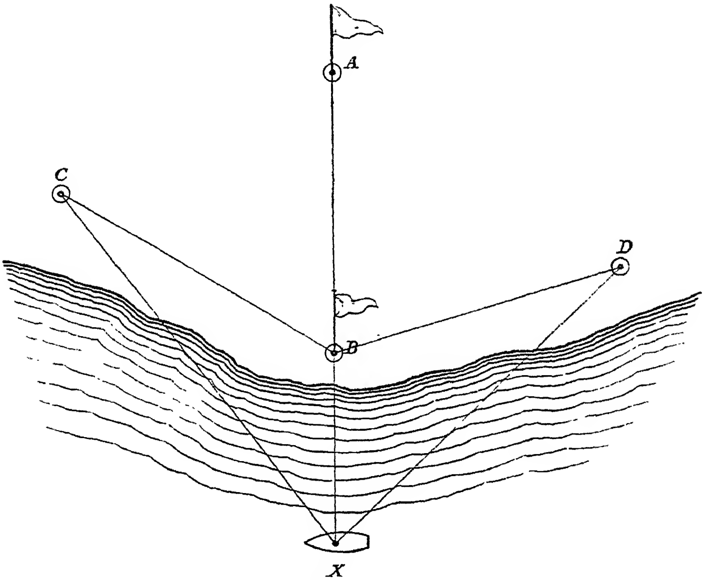


Fig. 109.

soundings the surveyor is wholly independent of the shore (excepting as regards the determination of the water-level), and he can stop afloat as long as the weather is suitable. It is true that without a rope or guiding-lines set out on shore the soundings cannot be quite so neatly spaced as with them, but on the other hand far more can be taken in the day. There should be no difficulty in distributing the soundings over the area in a manner that will delineate the bottom properly, and to this end the surveyor should take with him the means of plotting some of the points, such as the termination of lines of soundings. A second boat with

an anchor and cable is useful to mark the position of termination lines, so that a new line may be taken up.

In deep water, when the highest accuracy is not wanted, it is not necessary to observe at each sounding. The boat may be rowed with a steady stroke, and a sounding taken at every tenth stroke. At the tenth sounding a heavy lump of iron attached to a rope is dropped to the bottom close to the position of the sounding. The boat is steadied with the oars, and the oars worked so that the rope is nearly vertical. The necessary angles are observed, and the iron weight is taken up and the boat is started again in the desired direction. The boat best suited to such work is a light four-oared gig, which can be easily started and stopped.

It is not a bad plan to work parallel to the shore, keeping the boat approximately in some constant depth of water. The soundings will then follow approximately a contour line. Any slight displacement of an individual sounding will not matter so much as if the line were taken at right angles to the shore.

It is rarely necessary to calculate the position of soundings trigonometrically.

The construction for plotting the three-point problem has been given under minor triangulation.

For plotting bearings a station-pointer is sometimes convenient.

It is rather heavy, and if much work has to be done with it the paper is apt to be injured and made dirty.

A piece of tracing-paper and a good protractor affords a ready means of plotting the bearings observed to three or more points. Lines are ruled on it meeting at a point and making the observed angles with each other. The paper is applied to the plan and moved about until the lines coincide with the three fixed points, when the intersection of the lines will mark the point sought, which may then be pricked off.

It is always desirable to observe to four points; firstly, for a check; and secondly, because the intersections from three points may be ill-conditioned. If the boat is on the prolongation of some known line two angles suffice.

The ordinary pocket sextant is hardly a desirable instrument for sounding work. The mirrors are rather small, and it is therefore difficult to pick up distant objects quickly. An ordinary 8-inch sextant, as used in navigation, is preferable. The Admiralty provides a special sextant for sounding purposes, with 6-inch radius and extra large mirrors.

CHAPTER IX

RAPID SURVEYS

General Remarks.—The methods of conducting the field work of surveys hitherto described have been based on the assumptions—(1) that considerable accuracy is aimed at, (2) that ample time for the work is available, and (3) that a staff of trained assistants is to hand.

It will sometimes be found necessary to dispense with some or all of the above conditions, and to produce, in a very limited space of time, and without any skilled assistance, maps of areas (or of linear work, such as projected roads, railroads, etc.), with a limited degree of accuracy, but nevertheless which shall not fall short of what may be expected in small-scale surveys.

As with military reconnaissances, which vary in degree of accuracy from the roughest sketch to a near approach to a regular survey, so with work of the nature now referred to, the method to be adopted must depend on the object with which the survey is made, the time and party available, and the amount of accuracy of detail and of finish to be aimed at.

The methods to be adopted for lengths may vary from mere pacing upwards, and angles may be read by compass, sextant, or theodolite. There should, of course, be agreement between the accuracy aimed at in the two kinds of measurement. For instance, one would not measure angles by theodolite and distances by pacing. The principles on which the length of line may be found so that the error in plotting any one line may not exceed a given distance on paper have been fully explained in Part I., under the head of "Limiting Lengths of Offsets in Chain Surveying."

Levels.—Levels may be found by barometer, hand level, clinometer, or other methods.

The Tangent Scale Clinometer.—Various forms of clinometers and levels have been used for laying out approximate contours to guide the eye when sketching. In the Survey of India Department a convenient portable form of clinometer is used in conjunction with the plane-table. It consists of a base bar and plate,

fitted with means for levelling, and mounted on three ivory pins, for easy movement on the board. The base bar carries two vanes, whereof the back or sighting-vane is fitted only with a small sighting aperture. The forward vane has a central vertical slit, and is graduated on one side thereof with a scale of angles, and on the other side with a scale of *tangents*, the zero of both scales being level with the sighting aperture when the instrument is levelled. By means of this instrument the surveyor can measure the relative height of any object in view with regard to his own position, with the aid of only a simple calculation done on the spot. He can either deduce the height of his own position by observations to a fixed point of which the height is known, or to two or three such points for greater accuracy, or if the height of his position is known he can obtain the height of any other point in view within reasonable distance. It is necessary, of course, to know the distance of the point to be observed, and this (which should not exceed three or four miles) may be measured off from the plane-table. To obtain the difference of height between his own position and any other object in view, the observer looks through the hole in the sight vane, after levelling the clinometer, and notes what figure on the tangent scale is cut by the ray to the object. This figure, multiplied by the distance in feet, gives the difference of height in feet between the observer and the object.

Having found the height of his position, the surveyor proceeds with the clinometer to determine the heights of a number of points in his vicinity, such as the junctions of streams, prominent knolls, points on spurs where there is a marked change in slope, etc., entering their heights at once on the board. If the slope on which he stands is fairly uniform he can then find the position of points on the contours above and below him by observing the slope of the hill with his clinometer, and obtaining the horizontal equivalent for the difference in height between his position and that of the contour from the height indicator. The distances of the contours next above and next below the observer's position being thus determined in several directions, the contours can be easily traced between the points so found, and the operation is then repeated at subsequent fixings, or if desired, the surveyor can set up his plane-table on the contour by finding his position at one of the points on it, and using the clinometer as a level, lay out the contour in the usual way. Intermediate contours are then interpolated by eye, with the aid of the heights already fixed, at sufficiently close intervals to show all important features. Contours at 25-feet intervals are found suitable on surveys of 4 inches = 1 mile, and 50 feet on 2-inch surveys.

In surveys on smaller scales, it is impossible to bring out the hill features in satisfactory relief by contour lines at uniform vertical intervals. The relative steepness of slopes can be more effectively shown by judicious freehand, horizontal or vertical form lines, or hachuring. Numerous heights should be fixed by the clinometer to supplement those given by the triangulation, and these are especially necessary at obligatory points required for engineering works, such as junctions of rivers, bridges, passes in ranges, etc.

Rapid Triangulation.—The safest basis for all survey work is, of course, triangulation, and rapid triangulation, though not so accurate as regular operations, can (more often than is generally supposed) be carried out with satisfactory results—as has been done, for example, in the boundary operations in Afghanistan, and on the borders of China and Siam.

Such triangulation always proves most useful. It enables the topography to be executed with accuracy with the plane-table, by interpolation from the triangulated points, and the positions of these points being obtained by computing their latitude and longitudes, they can be plotted easily and accurately to any scale on any sheet.

In triangulating for frontier surveys, reconnaissances, etc., smaller instruments are used than on regular work, the bases cannot be measured with the same refinement, there is less regard for symmetry in laying out the triangles, and the three angles of each triangle are not always observed. It will often be found impossible to visit hills beforehand, to clear them and erect signals, so that natural marks such as rocks, trees, etc., must be observed to.

Having fixed a number of points by a rough triangulation, further details are filled in by traversing in the usual manner, always closing when possible on the previously fixed points.

In wooded or flat districts triangulation is often impossible, and then traversing alone must be relied on, but in this case the sketch of the area should be built up, as it were, piece by piece, from a part to the whole, and whenever possible forward points should be fixed and closed on.

Theodolites used in India for Frontier Work.—In India, the 6″ transit theodolite is found to be the most useful for frontier work. It is fitted with a "micrometer eyepiece" for subtense work, and with the level fixed to the vertical vernier bracket and *not* to the telescope. This theodolite is packed in two boxes, for convenience of carriage in hilly countries. Where it is difficult to carry a 6″ theodolite a 3″ has been found useful, and when

carefully manipulated gives good results for triangulation and astronomical observations. The stand of the theodolite is a folding tripod in which the heads of the legs are capable of being screwed so tightly that the whole stand becomes absolutely rigid, and the feet should be so shod that the shoes cannot work loose and cause unsteadiness.

Station Marks and Signals.—When the hills for forward stations have been decided on, and while the base measurements are being executed, men should, if possible, be sent ahead to clear the summits, when forest-clad, and erect signals on the highest points. In Afghanistan this was seldom necessary, for besides the hills being as a rule bare and rocky, their summits were generally marked by a cairn of loose stones, erected in pious memory of some saint. Occasionally there were several such cairns on a peak, and then they were all observed and the most suitable one chosen when the hill was visited. The cairns were easily removed to make way for the theodolite, and were carefully rebuilt when the observations were concluded. Before removing the cairn its centre should be noted. This may be done by laying a couple of lines of stones whose prolongations intersect in the centre of the cairn.

On the North-East Frontier of India and in Burma, where bamboos are obtainable everywhere, the signal invariably used consists of a bamboo tripod 15 or 20 feet high, of which the base forms an equilateral triangle with the station as centre. The poles are buried about 3 or 4 feet in the ground, and the sides of the tripod are covered with bamboo mats to within 6 feet of the ground. A basket in the shape of a double cone connected at the bases is also formed by splitting a bamboo up for about 3 feet of its length at the smaller end and working the latter into the required shape by means of strips of cane or bamboo. A long narrow flag may be advantageously used in place of the basket. This basket or flag is firmly secured to the tripod and centred over the station mark, which may be a circle inscribed on a rock *in situ* or on a stone sunk in the ground. The inside of a bamboo is white, and retains its brightness for some time when split. This side is placed outwards in the mats and basket, and the mark shines brilliantly in sunlight like silver. It is visible for very long distances, and sometimes when the hot weather has set in, such signals can be made out several miles away, glistening in the sunshine when the haze has blotted out the hills on which they stand. A single tree should always be left standing near the highest point when a hill is cleared, and its position with reference to the station carefully ascertained and noted in the "angle book." This assists the observer in recognising his station, and should the

signal be blown down, or destroyed by wild animals or otherwise, the tree can be observed to. In very exposed situations, a couple of trees or even a small clump may be left. Sometimes, when a hill is very narrow and rocky, it may not be possible to erect a signal, and in such cases two trees should be left standing and both observed to. When a hill has not been cleared or marked in any way, it is best to observe to the highest point or to any conspicuous objects, bushes, rocks, etc., but these may not be recognisable from other hills, or even when visiting the hills on which they stand. If several points in the same hill are of the same height, or seem equally suitable for stations, they should all be observed. If a surveyor cannot communicate with others working near him, and has reason to suppose that certain hills have been utilised by them as stations, though no signals may be visible, every point on each hill on which the theodolite could have been set up should be carefully observed. By following this practice during the Afghan campaign, when intercommunication between the Khyber and the Kuram columns was at times impossible, the triangulation carried on by the surveyors with each column was connected, and eventually computed satisfactorily, the correct angles being found during the computations by elimination, or by consultation among the observers, when they were able to compare notes.

Lengths of Sides.—The lengths of the sides of the triangles vary according to circumstances, such as the height of the hills, their configuration, the clearness of the atmosphere, the nature of the signals or objects observed, and the maintenance of symmetry, as far as possible. As a rule, the sides are from 10 to 20 miles in length. In Afghanistan is was possible to have triangles with 30 to 40 miles sides, and peaks were intersected at a distance of 100 miles. This was also possible during clear weather in the Shan States on the borders of Burma and China.

Observations.—In observing from a station, if time is limited, the surveyor must use his judgment as to what to omit, remembering always that the first object of the triangulation is to assist the topography. A good rule is to observe everything that is observable. It is impossible to say what will not be useful, and even a single ray to the most distant peak, village, or tree may prove very valuable as a check on topography to keep it in its place, and if possible all these rays should be laid down on the plane-table. It is sometimes the practice to set up the theodolite by the compass, so that the telescope points to the magnetic north when the horizontal limb reads 360°. By this means all readings are also magnetic bearings, and it frequently assists the finding of

a point which has been observed, to know its magnetic bearing from a previous station. Moreover, unless the bearing of even so plain an object as a heliograph, is known, it is possible in hazy weather, or when the carelessness of the heliographer prevents a full light from reaching the observer, to pass it unnoticed in moving round the telescope, however slowly.

Another advantage claimed for making the telescope point to the magnetic north when the theodolite reads zero, is that the variation of the compass at any station may be found by comparing the computed azimuth of any ray with its magnetic reading.

Vertical Angles.—For vertical angles both verniers should be read, with at least one change of face *in all cases*. Horizontal and vertical angles to stations should *never* be observed simultaneously. Vertical angles to all stations more than four or five miles distant should be observed at about the time of minimum refraction, say between 2.30 and 3.45 p.m. For intersected points the time is of less consequence, and so long as the observations are not made in the early morning or late evening hours, the observer may use his own discretion about adhering to the rule. If vertical angles be observed at an hour far from that of minimum refraction, observation should be made to some known station at the same time, for, the height of this station being known, the refraction at the time of observation can be computed by means of the vertical angle then taken to it. When possible, observations should be reciprocal, *i.e.* if A has been observed from B, B should be observed from A, as the effects of refraction are thereby eliminated, and a coefficient of refraction obtained for reducing single vertical observations. The height of the instrument and signal above the station platform must always be recorded in the "angle book," and in observing vertical angles to intersected points the particular part of the object intersected should be noted, as well as its height above the ground-level.

In observing to the tops of trees, or to signals in trees, the level of the ground is the height required, and if it can be seen it should be observed also in the case of intersected points. It will be impossible otherwise to measure the height of the tree, but from an observation this may be computed and the correction applied to all other vertical observations, from stations where the base of the tree is not visible. All circumstances likely to affect observations should be noted against each angle in the "angle book," also where any angles are exceptionally good or doubtful. This will assist the computer in apportioning the triangulation errors.

Recognition of Stations, Peaks, etc.—It is a good plan, in an unknown country especially, to make a panoramic sketch of the

intersected peaks as they appear in the telescope, marking carefully their outline and any peculiarity for identification. This sketch may be made in the "angle book" or on the plane-table. The peaks are numbered in the order in which they are observed from the first station, and the numbers adhered to throughout. Names are not always obtainable, but when they are they can be entered against the numbers for use in the final map. A triangulation chart should always be kept up, either stretched on a plane-table or capable of being pinned to one. This chart will greatly facilitate the work, and enable the surveyor to identify peaks which he would otherwise find it difficult to recognise. Mountains assume such wonderfully varying outlines from different points of view, and a broad square hill seen from the front will often become a sharp cone in profile, *e.g.* the rock of Gibraltar. Again, a hill presents quite a different appearance when viewed from below to what it did when seen from neighbouring hills, but when fixed on the plane-table there is no doubt about a peak. When, therefore, points have been observed to with the theodolite, the earliest opportunity should be taken of fixing them on the plane-table. If one or more rays have been taken to a hill on which a surveyor finds himself, but which he is not quite able to identify, he can ascertain his position by re-section, and if it falls on a previous intersection, or even on a single ray, it is probably correct. All other points around can then be identified. Again, suppose A and B (Fig. 110) are two fixed stations, and from A two hills, C and D, have been observed, but it seems doubtful if they can be readily recognised from B. It would probably be easy to find a plane-table station near A, say E, from which to fix C and D with sufficient accuracy to ensure recognition from B.

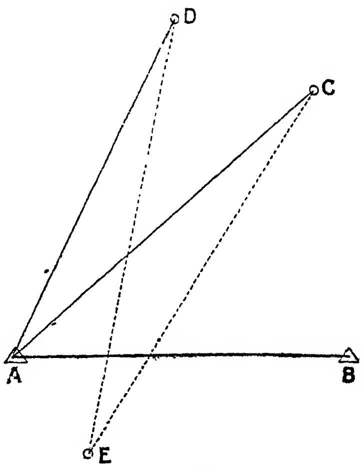


Fig. 110.

Description of Stations in Angle Book.—All stations of observation must be very carefully described in the angle book at the end of the observations made at the stations. This description of each station includes the situation of the hill, the exact

position of the station mark with reference to the single tree left standing, or some other well-defined natural object, the directions and distances of neighbouring villages, the local name of the spot (if any exist), the village in whose lands it is situated, the best way of reaching it, and any other information likely to assist any one subsequently wishing to make use of the station. All this is very necessary, especially in forest-clad districts, where the villages are constantly shifting, and where the jungle growth speedily removes all trace of village sites, fields, and paths, even as though such had never existed.

Stations in Forest-clad Country.—Forest-clad country such as is found on the N.E. frontier of India and Burma, the forests of the Gold Coast, etc., presents the greatest difficulties to the surveyor, and considerable expenditure of time and money and much labour are required to clear the hills for stations. In such cases the stations selected should, of course, be as few in number as possible. Sometimes, as in parts of Assam, the low undulating country is of such a general dead level that little is gained by clearing, or the hills are so broad and flat that complete clearing is out of the question. It then becomes necessary to raise the theodolite above the level of the surrounding country, sufficiently to obtain a clear view with a small amount of lopping of the branches of the neighbouring trees, and perhaps a narrow ray cut here and there to stations. The following method was employed by a survey party working in Assam. A suitable tree was selected, its trunk cut off at a convenient height 30 to 45 feet above the ground, and a scaffold erected around it. On this, and at about $4\frac{1}{2}$ feet below the top of the trunk, a platform of bamboo was made for the observer to walk on, the platform and scaffold being quite isolated from the tree-trunk. The theodolite, a 6-inch, was set up on the tree-trunk, dispensing with the stand. The highest point at which the theodolite has been used in this way, with perfectly isolated platform, is 45 feet. Some cotton trees with bare straight trunks rise to 80 feet, and platforms have been constructed at that height in these and other trees for plane-tabling, but they are not steady enough for theodolite work.

Action, should Triangulation break down.—When triangulation has been carried some distance, there is a chance of its breaking down, owing either to the configuration of the country, the non-identification of previously observed points, or circumstances which prevent flanking stations from being visited. When this is the case, if a station is visited before 9.30 a.m. or after 2.30 p.m., an azimuth of the sun, if visible, may be observed and referred to one of the back stations. This azimuth, with the

observed angle between any two well-fixed points, not too much in line, will give a very fair fixing of the position of the visited station, from which other points may then be observed. Interpolations play a more important part in rapid work than in regular survey, and stations may be made in the middle of a valley where there is no conspicuous object to which it is possible to observe.

Suppose A and B (Fig. 111) are two fixed points on a range parallel to the route being followed, D being a camp on that route, and that a peak C on the range AB has been observed from B. The distance AB and azimuth are known. The surveyor observes the azimuth of A or B from D, and the angle ADB. With these data the other angles of the triangle ADB may be found, and the position of D computed. From D, C may be observed and fixed. C must then be visited and another forward hill E observed. From C and a further point F on the route, the hill E is similarly fixed; and so on, thus triangulating along a single ridge, but without observing two points on the flanks. With care, this method is susceptible of considerable accuracy.

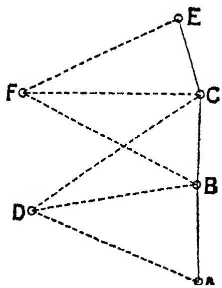


Fig. 111

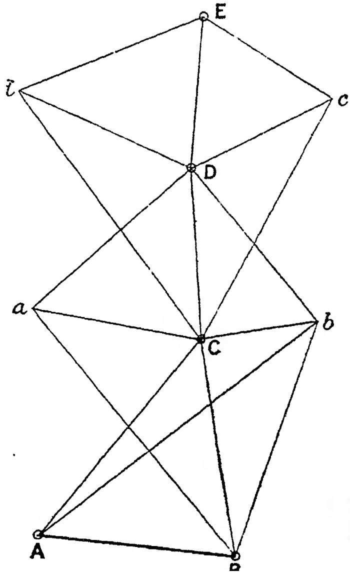


Fig. 112.

Again, let B, C, D, E (Fig. 112) be points along a continuous ridge, and *a*, *b*, *c*, *d* peaks on the flanks which cannot be visited, AB the last base fixed by triangulation, from which points *a*, C, *b* have been fixed. Proceeding to C, angles are observed to all the points. Then in the triangles CB*a*, CB*b* we have the base CB and the two angles at the base in each triangle.

We can then find the third angle, and determine the bases Ca, Cb. Visiting D we observe to C, a, d, E, c, and b. In the triangles CDa, CDb, we have the angles at C and D, and can therefore deduce those at a and b, and with bases Ca, Cb we get values of CD, from which base c and d are fixed, and so on.

Action when Triangulation is interrupted.—If the triangulation be interrupted, a careful traverse checked by astronomical observations should be carried on till the triangulation can be started again, when, from a fresh base, it may be possible to connect the new work with the old. If, for instance, a pass has been crossed, and the surveyor finds himself unable to see more than one fixed peak (X) and cannot ascend any hills in the neighbourhood, he could measure a base and observe to X from this base, and also the azimuth of X from either end. The values for the base can then be computed, and fresh forward triangulation started.

When the route approaches north and south, latitudes should be observed at all camps, and azimuths observed to all peaks in the line of march, or nearly so. When camping abreast of any of these peaks, their position must be fixed with reference to the camp, by means of a short base with the usual latitude and azimuth. The difference of longitude can then be easily computed.

A Series of Azimuths necessary in Rapid Triangulation.—Of course, in all rapid triangulation, as will have been gathered, a continuous series of azimuths is essential, and this involves the computation of the latitudes and longitudes of all stations and intersected points. Computations should always be kept up to date if possible. If the latitudes and longitudes of points required for use are calculated, these points can be more accurately plotted by their co-ordinates than by distance and bearing. The distance between points not directly connected can be readily computed, and this is particularly useful for interpolations. Again, latitudes and longitudes are easily communicated to other surveyors by telegraph, heliograph, or letter.

Triangulation and Topography by One Surveyor.—If triangulation and topography have to be performed by one person, he should consider carefully how much attention to bestow on each. The objects of the survey will assist him in coming to a conclusion. Triangulation by itself is useless, and unless it is necessary to know the positions of certain points with accuracy, or unless the triangulation is intended to assist other topographers, it should be subordinated to the topography, and carried out only so far as it assists the execution of the work actually in hand.

Heights in Plains and Valleys.—Some surveyors neglect to fix any heights in the plains or valleys, only observing to hilltops.

Of course, the heights of as many hills as possible should be fixed, not necessarily to be entered in the final map, but to assist the topographer to interpolate heights in valleys and open ground. But it is also important to observe vertical angles to well-marked points of features in low ground, such as conspicuous bushes or rocks on the banks of a river, the junction of two streams, pagodas, temples, the most prominent house in a village, etc. The distances for computation may be taken from the plane-table. It is impossible, in maps on a small scale, to give by any system of hill shading an accurate idea of the relative heights of neighbouring valleys with intervening ranges; but in consulting maps, whether for movement of troops or engineering operations, one is materially assisted in coming to a right judgment by a few figured heights in the low ground, as well as on the range between. Again, passes should always be observed to. It is often more important (though this is not so often realised) to know the height of a pass or the saddle between two peaks, than the heights of the peaks themselves.

Height of Base Measurement for Reduction.—In measuring a base, if there is no reliable height for it, an approximate height must be obtained from barometric observations and used throughout the triangulation. When eventually a trustworthy value for the datum is arrived at by connection with a trigonometrical series, or to the sea, etc., a constant correction can be applied to all the heights. In Afghanistan, barometric heights were found to be very unreliable, even those obtained from a George's mercurial barometer being unsatisfactory. It was found that the temperature correction as given in text-books is excessive, and it is suggested that this is because in that country the temperature observed at the ground-level is due to a highly heated stratum of air, giving no clue to the temperature of the column of air immediately above the place of observation.

Among aneroids the smaller patterns will be found to be quite as accurate, and much more convenient to carry about than the larger. Very satisfactory results have been obtained from boiling-point thermometers, if three at least are used, and the mean taken. Aneroids should always be read immediately on arrival in camp and again next morning before starting, the changes due to climatic variations of pressure being thereby eliminated to some extent. During halts a series of careful hourly readings should be made to get data for correcting observations for the diurnal wave of pressure. Whatever instruments are used, no opportunity should be lost of comparing the results with the heights obtained by triangulation. Such valuable opportunities occur while waiting on a hill for it to be cleared or while observing, and barometers may be

hung up in a convenient spot and read at intervals, while observations are also made with the hypsometer.

Watches.—Watches form an important item in an explorer's outfit. Chronometers are unsuitable for work on land, as they do not stand even the most careful method of carriage. Keyless half-chronometers have been found to give exceedingly good results. An explorer should have at least three of these with him for comparison, and care should be taken to keep them always under the same conditions of position, temperature, etc. Many watches vary greatly in rate when changed from a horizontal to a vertical position, when lying on their face or the reverse, or if the ring is to the right or left of the watch. Jolts should be carefully guarded against, and a vertical position may be secured by having special pockets in a coat and keeping the watch hung up in the coat at night. Great changes of temperature should also be avoided by wrapping up the watches in cloths. It is well to carefully pack your watches yourself, and the writer has found a despatch-box very suitable, as it could be fitted into a light bamboo sort of chair and carried on a man's back. The box was simply lifted from the table into its chair, and back from the chair to the table, and the watches always remained in the same position either on the march or when at rest.

Before starting, the rates of the watches should be ascertained as nearly under the conditions of travel as possible. During halts the rates should be checked by observations for time, and the errors determined whenever the longitude of a place is known.

Precautions should be taken against damp, getting wet in crossing streams, or accidents, by adapting waterproof covers or by waterproofing the case as described in "Hints to Travellers."

Plane-table Topography

The Plane-table.—The plane-table is without doubt the most accurate topographical instrument we have, and one of the most useful to explorers. It enables the triangulator to recognise his points, and when necessary graphic triangulation can be carried on successively side by side with the topography. For instance, two officers of the Indian Survey Department, working in 1872–3 on the North-East Frontier of India, were able to carry a series of triangles with their plane-table through the Naga Hills to Manipur, erecting the usual marks (but without astronomical checks) during a rapid march of 120 miles, and when this work was afterwards incorporated in the regular triangulation very little correction had to be applied anywhere, the last station in Manipur being only out

of position by 2300 yards in longitude, and less in latitude. It was done by each taking a side of the series daily, and besides plane-tabling, these officers managed to observe a number of angles both of the triangles and intersected points, the computations being worked out after they reached Manipur.

The plane-tables in use in the survey of India are similar to those described in Chap. VII., Part I., being of a simple and strong construction. The general size is 24 by 30 inches, although on some of the Frontier expeditions a smaller and lighter one, 20 by 24 inches, was used with advantage. The writer has seldom used the latter, as he has found it possible in most cases to carry the larger size—the advantages of which are obvious, especially if the operations are likely to be extended over an area greater than the field of the small board. Sometimes, also, it is not possible to foresee exactly in which direction the work may eventually lie, and a large board enables the surveyor to face a change of front with equanimity. Again, on a large sheet very much may be done in the way of tentative geography, "guesses at truth." The writer has always sketched in lightly in pencil what seemed to be the probable geography of all distant country embraced by his board and beyond the limits of ground which could be accurately delineated. He has found it exceedingly interesting, as the work progressed, to watch these guesses turn into facts, or, if wrong, to see where, how, and why he was misled. It is an excellent education in reading country, and even a "guess at truth" is always better than an absolute blank on the board.

Suitable Scales.—The scale to be adopted is decided by various considerations, such as the objects of the survey, the amount of detail required, the area embraced, the time allowed, etc. For trans-frontier surveys and explorations the scale adopted by the survey of India is 1 inch = 4 miles for general work, special portions being surveyed on a larger scale where necessary. On the small-scale maps portions of country may be generalised which would have to be altogether left blank on a larger-scale map requiring more attention to minor features, and distant triangulated points may be plotted which materially assist the plane-tabler. The engineer in charge of the operations, if he had a large party, would find it a good plan to have a small-scale board embracing the whole area of operations, on which he could note the progress of the work, compiling it from the tracings sent in by his assistants, and checking it in the field from time to time.

The methods of "setting up" the plane-table and finding the surveyor's position thereon from certain fixed points have been fully dealt with in Chap. VII., Part I. These are the same in

principle for all plane-table work and only a few remarks on this head are here necessary. Before beginning a reconnaissance a surveyor should (when triangulated points form the basis of his work) start it from the nearest station, setting up and levelling his table over the mark stone. In hilly country it is very necessary that the table should be level, for if it is not, where some points tower above the observer, while others lie far below him, the fixing will be inaccurate. The board is now oriented by placing the sight-rule on the line joining his station with the most distant visible fixed point plotted on the board, and turning the table round in azimuth till the sight-rule intersects that distant point. All the other trigonometrical points on the board can then be tested, and if any are found to be wrongly plotted they should be carefully compared with the computations. It may sometimes happen with intersected points that by some accident, *e.g.* two hills with similar outlines having been observed as one and the same hill, the computations give apparently satisfactory results, and a point is plotted which cannot be identified. When obviously wrong it should be at once erased, but if there be any doubt a query may be placed against it till the question is finally decided.

The method known as "setting by the back ray" is the best for setting up a table in true azimuth, as it is independent of abnormal variations of the compass. The other method is by interpolation, or resection (as some prefer to call it), and this may be done by compass and the intersection of the rays from two trigonometrical points, or without the compass from three such points ("the three-point problem," *vide* Part I.), but in neither case is there any test of accuracy. In the first, there may be some abnormal magnetic variation, and in the second, if the surveyor's position falls on or near the circumference of a circle passing through the three points, no satisfactory fixing is possible. It is therefore better that the surveyor should, if possible, fix himself from four triangulated points at least, trusting to his compass for obtaining an approximate position only. Positions fixed by intersection where the angle is not less than 60° may be regarded as correct, but where the angle is less than 60°, as only approximate. Of course an excellent fixing may be obtained if several rays from A, B, C intersect in *b* at an acute angle, and another ray from D can be obtained passing through the point of intersection and nearly perpendicular to the line bisecting the angle A*b*C at that point.

Each "plane-table fixing" should be distinctly marked with a red dot for future reference. Names may be written on the plane-

table as they occur, or numbers may be inserted, a list of the numbers and the names to which they refer being made at the side. Numbers do not interfere with the sketching as much as names do, especially on a small scale-map, but care must be taken that they are not obliterated as the work progresses.

Among the mountains, where the sun is high and casts but few and small shadows, it is often difficult to distinguish under features or to separate near and low ridges from higher ranges rising behind them. At dawn or sunset, however, the ranges stand out distinct from each other, and when the shadows are long spurs and ravines reveal themselves on what, in the bright light of day, appears to be a flat and continuous mountain face. In the Far East white mists lie like a sea of cotton-wool over all low ground and in every valley from early morning till 9 or 10 a.m. These soft clouds mark out each spur and ravine, materially assisting the surveyor in following up the courses of the principal valleys with those running into them, and distinguishing between the large and small affluents of the main streams. A re-entering angle in the hills, which during the day seems to be only a deep ravine, is now seen to be the mouth of a long valley, and what seemed to be the opening of a wide valley is discovered to be only a short ravine.

In the early spring in the Shan States and on the borders of China and Siam, thick weather obscures all distant landscapes, and a brown haze like a curtain shuts out everything four or five miles away on either side of the traveller's path. Under these circumstances survey parties are limited to plane-table traverses, working at times a good deal on "the back ray" system, intersecting points on either side of the route, and filling in as much of the topography as is visible. Native surveyors can bring in a good deal of creditable work done in this way, the distances being generally measured by pacing. Pacing may be done very accurately, and the writer has worked with some Sepoy surveyors who could (except in very difficult ground) pace to within a quarter of a mile of the truth in a march of 15 to 20 miles. Actual traversing with the plane-table is slow work and troublesome, especially where jungle or the nature of the country prevents the path from being visible for any distance at a time. In such cases a compass traverse may be carried on, the bearings and distances being entered in a book, and at every 2 or 3 miles, where an opportunity for sketching the country occurs, the traverse is plotted on a large scale and reduced to that of the topography, and from the position thus found on the plane-table the topography is worked in.

Plane-tabling and Traversing with Prism Compass combined.—Where two men are available, one to traverse, the other

to plane-table, *more* and *better* work can be done. When travelling from Hunza through Wakham to the Dorah Pass, the writer was obliged to do long marches averaging 19 miles a day, and continuous triangulation was impossible. He had a very good Mahratta surveyor, to whom he entrusted the plane-table, while he ran the traverse with a subtense compass and 10-foot rods as subtenses. The variation of the compass was frequently ascertained. The Wakham valley is fairly open, a river running down the middle and high bare hills, often snow-capped, rising on either side. The country is favourable for traversing, and distances averaging over half a mile could be measured, the rate of progress being about 2 miles an hour.

Personal Description of a Piece of Work done by the Writer. —(*The following is given as described by the writer, the late Major-General Woodthorpe, C.B.*) "Two men with 10-foot rods were employed. At starting one of these men (A) was left at the camp, and walking as far as I could without losing sight of him, I set up my compass and sent the second man (B) on to the farthest point B_1 of the road visible from my station. (As a rule the subtense compass is not accurate beyond a mile, and this limited the distance along each bearing in traversing.) While he was moving there I observed back to A, took the bearing and distance, noted them in a book, and signalled to him with flag or heliograph to come away. A round of bearings to all conspicuous peaks was then taken, and by this time B had arrived at B_1. He was observed to, and dismissed by signal. He then marched on to B_2, leaving a mark at B_1 for A, a small stick with a piece of red rag attached, a branch torn from a bush, etc. Arrived at B_1 A halted. I had reached my second station, and B was nearing B_2. The same procedure was adopted, A and B were observed and a round of bearings taken, A advanced to B_2 and B to B_3, and so on. Before leaving camp my surveyor set up his plane-table at A, drew rays down the valley and to all the peaks, and commenced sketching. He then hurried on to overtake me, and I plotted on his board the traverse up to that point, with the intersected peaks which fixed his position there, and he continued his sketching. Having thus obtained his position from my traverse two or three times, he himself had intersected enough points to enable him to resect his position independently of the traverse for a time, and his work thus checked the traverse, as the traverse in its turn checked his work." Of course, when traversing in the hills with a subtense compass, it is not necessary to keep to the road, for longer distances can often be got by going up or down the slope a little way on either side of the path. Besides, the compass and rods are thus

kept clear of the line of march, the baggage animals, coolies, etc., who otherwise much impede the work.

Use of Astronomical Observations.—As soon as the astronomical observations are completed, the latitudes should always be worked out, though this need only be done with sufficient refinement for the scale of work employed, in order to fix the position of the camp, and to check the traverse and the topography. An additional advantage in computing the latitude at once, is that if there is anything wrong about it, other stars may still be observed before it is too late. Although it is hard work and cruel sometimes, as on the Pamirs, to take stars when the thermometer stands at zero, and the observer has to warm his fingers over the candle between each turn of the screw, yet results well repay the trouble. All places where astronomical observations have been taken should be distinctly marked on the plane-table and described carefully for the use of future surveyors.

Perambulator Work.—Other means besides subtense measurements of distances in traversing have been alluded to, and these require only a few words. A perambulator is useful and convenient when the country and roads are favourable, but it must be very strong and therefore heavy, and is not very portable. A cyclist will understand the principle of the perambulator and also its limitations.

Chains.—A chain, even the lightest, is also subject to certain limitations, and cannot always be used.

Canes.—Canes were used in Assam when surveying forest-clad and swampy country, and where traversing could only be carried on along the numerous small and shallow streams. The bends were too short for any subtense methods of measurement. Fine light canes, in lengths of 50 to 100 feet and about 1·5 inch in circumference, are obtainable; their length is very constant, they float in water, thus giving no trouble to the chainmen, and are easily pulled taut. They also possess the advantage over chains and ropes, that they are easily drawn through dense scrub or undergrowth without being caught by thorns.

Traversing Rapid Streams.—In traversing rapid streams, it was sometimes impossible to use canes owing to the strong flow of water and frequent occurrence of deep pools.

Berthon Boat and Rate of Current.—A Berthon boat or "dug-out" (a country boat hollowed out of a single tree) may be used. Measure along the bank certain distances carefully, and then note the time taken in passing them by the boat when carried down by the current only. This gives its rate, and a prismatic compass, suspended in gimbals, standing in the boat like a ship's compass,

gives the bearings, and thus with a watch it is possible to make a rapid traverse of a stream with all its windings. With a fixed point to start from, and another to close on, this is a sufficiently accurate way of filling in such detail, especially as such streams are continually altering their course within certain limits, as the current undermines the banks or a fallen tree dams the stream, and it straightway finds a new channel for itself.

In many ways timing is a more convenient method of determining distances than pacing, as it is not so tedious, and there is less danger of error. It is so easy to lose count in pacing, and especially when with a column on the line of march, or with coolies on a narrow path, where slight checks are frequent. It is better then to measure the distances by timing, and the rates of the column will be a better one to take than one's own. Timing requires a good deal of attention, but between the observations of the watch one is free to note objects on the line of march, and to watch the topography, whereas pacing demands one's whole attention throughout.

Inking in Work.—It is difficult to lay down any rule for inking up a plane-table sheet. Some men can give an excellent idea of the country by brush work, and when done well this is effective, and besides names can be written over it without interfering with topographical detail, which is a decided advantage. As brush work cannot be done in the open, it is necessary to draw in all the features carefully in pencil on the spot with "form lines," and on the first opportunity cover them with brush work. Some men can only indicate features by "form lines" or hachures, but whatever method is adopted all inking up should be done as soon as possible, either every night or at each halt, while the topography is still fresh in the mind. Where villages are numerous it is necessary to show them by numbers only, or all under features will be obliterated by the names. The spelling of names in a foreign country or in one which has no written character, must be as accurate as possible, and on some authorised system to ensure uniformity.

In parts of a district which are only *approximately* surveyed, all streams should be shown by dotted lines and the features so indicated as to distinguish them from accurate survey work, and to leave no doubt as to its inexact character. It may be convenient to add to the plane-table survey work of a less exact nature than the surveyor's (such as rough native maps, routes performed by untrained or little trained assistants), and this should also be distinctly shown to be approximate or unreliable. In the early days of our occupation of Burma, several Sepoy surveyors and others surveyed routes lying outside the writer's work, but as they

generally started from some of his fixed points or known places and returned to others, he was able to adjust them to a certain extent, and they appeared in dotted lines on the first maps of the country. They were very useful, though the officers using the maps knew that they were not absolutely accurate. Where such routes are compiled on the plane-table sheet, notes should always be made to show whose work it is and how it has been performed, as a guide to the value and reliability attaching to it.

Photographic Surveying

General Remarks.—Since the year 1849, attempts have been made to use photography for mapping purposes, and in recent years many papers have been written on this subject, including a practical treatise by Mr. E. Deville, Surveyor-General of Canada, and published in 1895 by the Government printing bureau at Ottawa.

In the north-west of Canada many thousands of miles in area have been surveyed by metro-photographic methods with great success.

Owing to the great difficulty in obtaining good photographs in this country (showing distant details clearly), where the atmosphere is seldom clear for long together, this system has not found favour, but in our colonies such difficulties do not exist, and it is therefore deemed desirable to describe how such surveys are made.

The camera for taking the views should possess the following qualities, etc., viz.—

1. A good lens with an aperture of 45°.
2. A firmly constructed body (not of collapsing pattern) to take flat glass plates.
3. Levelling screws and levels to set the optical axis truly horizontal, and the picture plane and vertical rotating axis truly vertical.
4. Points, or a hair stretched, to mark on the negative the trace of the horizontal plane, and also the principal plane of the perspective view.
5. Some kind of sighting arrangement with which to direct the optical axis of the lens on objects.
6. The camera should be mounted in a vertical rotating axis, and have a graduated horizontal limb on which to read the angle through which the instrument is rotated.
7. Clamps, verniers, microscopes, and tangent screws, etc., as for a theodolite.

Cameras made of metal are superior for this purpose to those

made of wood, as there is then no fear of their losing shape or dimensions.

The instrument perhaps best known in this country, and therefore chosen for illustration, is the Bridges-Lee Photo-theodolite, made by Messrs. C. F. Casella & Co., Whitehall, London, but there are many other forms on the market, particulars of which can be obtained in books, such as Flemer's "Phototopographic Methods and Instruments" (Wiley & Sons).

Description of Instrument.—The Bridges-Lee instrument consists, as will be seen from Fig. 113, of a camera on a theodolite mounting, so that it can be levelled, and its axis turned through any desired horizontal angle from any given starting-point.

Above the camera is a telescope enabling the instrument to be used as an ordinary theodolite if desired.

At the back of the camera is a frame carrying two hairs, one vertical and one horizontal. By turning a handle at the side, after the slide is drawn but before the cap is removed, this frame can be moved outwards, so as to bring the hairs into contact with the photographic plate.

In this position both hairs will appear on the finished photograph, and their intersection should accurately coincide with the foot of the perpendicular let fall from the centre of the lens on to the photographic plate, which should be truly vertical.

Thus the hairs should mark the intersections of the plate with vertical and horizontal planes through the centre of the lens.

The same movable frame carries a transparent compass ring, graduated to half-degrees, so arranged that the compass bearing of each photograph is impressed photographically on the plate along with the hairs.

This gives a check on the readings of the horizontal circle, and assists in the identification of the photograph. There are also two small carriers for holding small celluloid plates on which any kind of notes, such as identification numbers, can be written, and these will appear on the photographs.

An example of the kind of photograph which results is shown in Fig. 114, in which, however, the vertical hair appears white instead of black, and the horizontal hair is not distinct.

Field Work.—Now, in Fig 115, let AB be the base line, known in length and position; 1, 2, and 3 points whose plans are required.

Let the camera be centred over station A, levelled, and the graduated circle set to read zero on station B.

This may be done either by bringing the hairs into contact with the ground glass screen, and then setting the vertical hair to coincide with the image of the signal at B (for which

purpose a magnifying glass is necessary), or it may be done by means of the telescope on top.

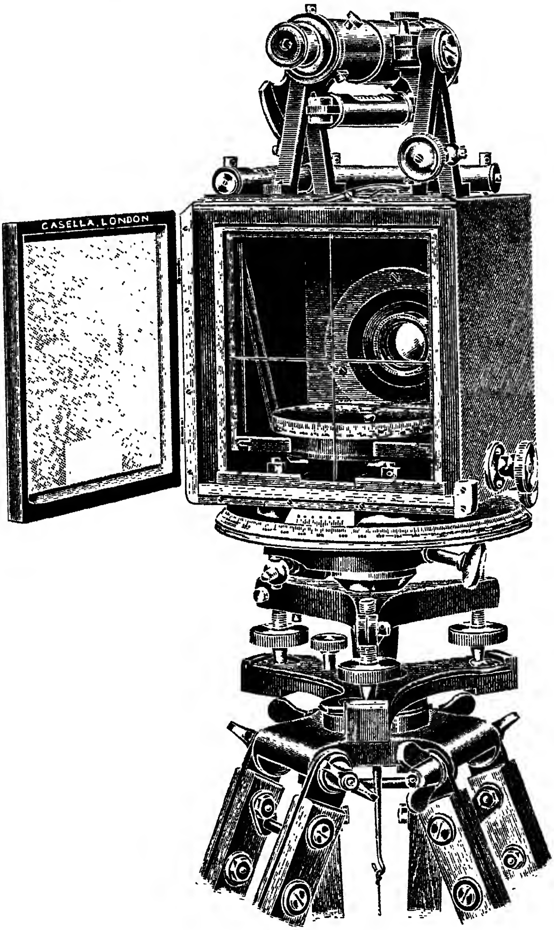


FIG. 113.

Now unclamp the upper horizontal plate, and turn the camera until its field embraces as much as possible of the area to be surveyed, and read the horizontal angle.

Let XAZ be then the axis of the camera. The angle BAZ will be known, and if X3′ be the position of the photographic plate, X should coincide with the vertical hair in plan, X3′ should be perpendicular to AX, and 1′, 2′, and 3′ will be the positions of the images of 1, 2, and 3.

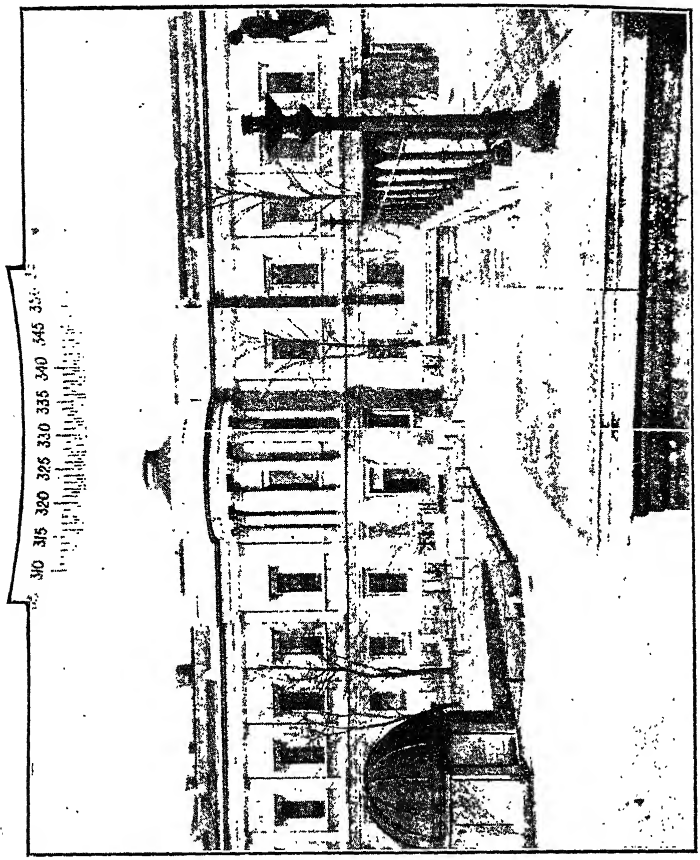


FIG. 114.

The distances X1′, etc., measured *horizontally* from the hair X to these images will be independent of whether the points are high up or low down, and they are all carefully measured off from the negative.

The line AB, A1, etc., are to be regarded as giving the *directions* only of B, 1, etc., from A. The principle is independent of the *distances* of the points, except that the focus cannot be altered, and if the points are so near as to be out of focus, the smallest stop must be used.

Plotting.—Now, on the *drawing*, suppose AB of Fig. 116 to represent the plotted base line on any scale. Lay off the known angle BAZ and draw the axis ZAX. Scale off AX to represent (full size) the focal length of the camera (which is supposed to be known), and draw X3′ perpendicular to XZ. Then set off (full size) from X the distances X1′, X2′, etc., as measured from the

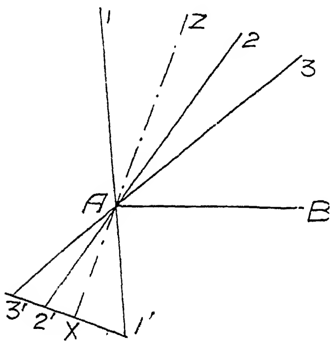


FIG. 115.

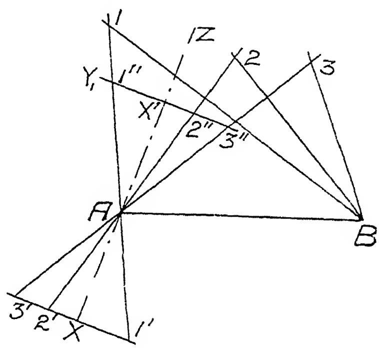


FIG. 116.

negative, and join 1′A, 2′A. Clearly the plans of 1 and 2 will lie somewhere on these lines produced.

Now, if another photograph of the same points be taken from B and treated in the same way, we shall obtain a new set of rays B1, B2, etc. (Fig. 116), on which the points must also lie.

Thus the plan of 1 is fixed by the intersection of A1 and B1, and so on.

Of course the lines AB, A1, etc., are shown on the drawing *to scale*, whereas AX and the distances X1′, etc., are *full size.*

As a matter of practice, the focal length AX is really set off along AZ itself, as shown at AX′ in Fig. 116. If X′2″ be made equal to X2′, but set off in the opposite direction, it is clear that the line A2″ should coincide with 2′A, and so on.

The advantage is that the lines have not to be produced so far, thus avoiding accumulation of drawing errors.

Sometimes the photographs are enlarged so that the line Y′X′ lies outside the plans of all the plotted points. Thus the rays

have not to be produced at all. It is, of course, a sound rule in all methods of plotting, that if a point is to be fixed by producing a line, the part produced should be short compared with the original line.

Plotting by Calculation.—Clearly if the angle X′A3″ be called β, we have $\tan \beta = \frac{X'3''}{AX'}$. Hence β can be calculated, as AX′ is known, and X′3″ is measured from the negative.

The value of β for any distance X′3″ can, in fact, be tabulated beforehand if desired. Hence as BAX′ is known, it is easy to find the value of the angle BA3.

Similarly the value of AB3 can be found from the photograph taken at B. Hence, knowing AB, we can calculate the distances A3, B3, and, if desired, the co-ordinates of the point 3, as in traversing.

Some such method should always be used for fixing new stations, though, if 3 be such a station, it is generally quicker to measure the angles BA3 and AB3 directly (using the telescope), with this instrument.

Levels.—If levels are desired as well as a plan, the level of the station—say A—and the height of the instrument above it must be known.

Let f be known focal length (= AX in Fig. 115),
x the horizontal distance of the image from the hair, as measured on the negative (= X1 in Fig. 115),
y the measured distance between the image of the point and the *horizontal* hair,
d the horizontal distance between the station and the point measured to scale from the plan,
and H the difference of level between the instrument and the point;

then clearly
$$H = d \times \frac{y}{\sqrt{f^2 + x^2}}$$

y, x, and f may be measured in any unit, but they must all be in *the same* unit. H will then be in the same unit as d.

H will be *plus* if the image of the point in the negative is *above* the horizontal hair, and *minus* if below.

All points whose images lie *on* the horizontal hair will themselves be at the same level on the ground. In other words, the line passing through those points on the plan will be a contour line at the level of the centre of the instrument.

But points whose images lie on a line *parallel* to the horizontal hair will *not*, in general, be at the same level.

Graphical Construction for Levels.—In Fig. 117, let A be the station, XY the plotted position of the negative, X1′ the measured intercept for a point 1, and 1 the plan of that point, all as already described.

At 1′ draw 1′*a* perpendicular to A1′, and make 1′*a* equal to *y*, the distance of the image above or below the horizontal hair.

Draw 1*b* perpendicular to A1, join A*a*, and produce these lines, if necessary, to meet at *b*. Then 1*b* gives the difference of level between the point 1 and the centre of the camera at A.

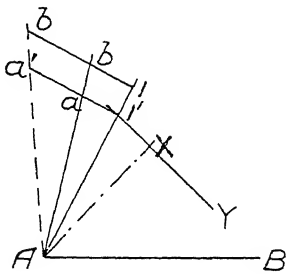


FIG. 117.

It will be measured to scale in the same units as the base line AB was laid off, assuming that 1′*a* has been set off in the same units as AX and X1′.

It is advisable, however, in most cases to increase the scale for 1′*a*, when the scale of 1*b* will be increased in the same ratio, as shown at 1′*a*′ and 1*b*′.

Some General Remarks.—In practice the writer has found it desirable, whenever time would allow, to make a rough sketch of the ground at the time of the survey. The making of such a sketch helps one to ensure that all the points which must be shown on the plan are visible from the stations selected. It is, moreover, of tremendous value in plotting, as the points can be numbered on the sketch instead of on the negative. Not only does this avoid the risk of obliterating some valuable detail on the negative, but one can tell by a mere glance at the sketch which are the *leading points*, as it were, of the survey, which we must at once look for and identify on the negatives. Finally, the sketch is of great value to the draughtsman when he comes to join up his plotted points. This sketch is seldom *completed* on the field in all details which are clearly visible, as these can easily be filled in afterwards from the photograph. But it generally saves time to complete the sketch in this way at the time when the negatives are being measured.

The method thus becomes a kind of sketch survey, in which some points are fixed by intersecting rays, as in plane-tabling, and then joined up.

The difference is that, given suitable ground, the points are fixed (so far as the field work is concerned) much more rapidly than is possible by any other method.

In this reduction of time in the field lies the special advantage of photographic surveying.

It is of comparatively little use in towns or in gently undulating country, well wooded, with high hedges, etc., such as is fairly common in this country. Each point which is to be fixed must be clearly visible from at least two suitably arranged stations; hence on such ground the multiplication of stations becomes tremendous.

The ideal kind of ground is where there are few trees or houses, and where the stations are on high ground overlooking the area to be plotted, or *vice versâ*.

So far as the experience of the writer goes, the method is rather more accurate than plane-tabling, and about equal to tacheometry if the leading stations have been previously fixed by triangulation.

If the country to be mapped has been triangulated, first obtain a plan showing the trigonometrical points, and if no such triangulation has been made, the surveyor must first take a theodolite and make one for himself. Even if not convenient as camera stations, the triangulation points so determined will serve as a means of fixing the exact position of the camera stations, as well as the orientation of the camera, when taking the views.

The aim in all cases is to obtain clear views of every part of the district to be mapped, from *at least* two stations, the ordinary requirements of any triangulation as to well-conditioned intersections being observed, since fixings by acute or obtuse rays are obviously bad. A kind of scheme should be evolved for the general location of camera stations in the district to be mapped.

Office Work.—The work to be done in the office consists in the preparation of the photographs from the undeveloped plates exposed in the field, and in plotting the topography with the aid of these photographs. Having carefully developed the negatives and secured good prints, the latter may be handed over to the topographer.

The exact positions of the photograph stations having been plotted on the plan, the topographer selects a suitable pair of pictures, containing in each a convenient number of salient points to plot. He next makes fine red-ink dots on corresponding points in each picture, writing a small number against each for identification. This work can really be best done by the field topographer, whose memory will aid him in identifying points in each picture. Unless very well-defined objects, such as churches, detached houses, conspicuous monuments, trees struck by lightning, etc., can be found in each picture, the writer has found considerable difficulty in making the necessary identifications, and especially if the stations have little relative command over the area photographed.

Mr. Deville, with his Canadian experience, however, states that this is not so, but a mistaken idea, since there is no difficulty whatever in identifying any number of points on moderately good photographs.

The plotter next takes two narrow strips of paper, as long as the picture (enlargement) is broad, and rules a fine line across the middle of each. On one of these bands he then marks off the exact distance of each dot on one of the views, right and left of the middle line, noting the numbers near each dot. These distances are then transferred to the trace of the picture plane. This process having been carried out with the view from another station, pins are driven at each station point, and two long fine threads, with loops at one end and a piece of elastic and a weight at the other, are attached at each station point. By placing each thread successively over dots with the same number in each trace, the intersections indicate the precise positions of each point successively, as indicated by each dot. At this stage the topographer can note if the intersections in each case are "well conditioned." The process above described can be repeated indefinitely with other photographs, from the same or other stations, and outlines between the points so plotted can be filled in as the work progresses.

The above method is that which has been adopted in Canada and elsewhere for plotting ground plans, and experience has shown that with reasonable care satisfactory results are attainable.

Developments of the Method.—There have been many comparatively recent developments in photographic surveying, such as the application of stereoscopic methods, etc. Many of these will be found described in Flemer's "Phototopographic Methods and Instruments," and elsewhere. Messrs. Aldis, of Birmingham (and possibly other firms), have made a special reflector by means of which it is possible to take a photograph all round the circle from a given station in one exposure. A scale of degrees is impressed upon the plate at the same time.

All these developments are beyond the scope of this book, and will not be further alluded to.

Tests for Instrument.—We shall now discuss the tests which have to be applied to such an instrument as the Bridges-Lee Photo-theodolite.

Of these, perhaps the most important are—

(*a*) to find the true focal length AX (Fig. 115);
(*b*) to find out if the vertical hair passes truly through the foot of the perpendicular from the centre of the lens to the plate, and if it, as well as the plate, is truly vertical;
(*c*) to find out if there is any appreciable distortion of the image;
(*d*) to find out if the horizontal hair truly represents the intersection of the plate with the horizontal plane through the centre of the lens.

Of these, (*a*) and (*b*) may slightly vary with different slides, so that photographs should be taken with each slide for the test, if a very accurate result is desired.

Methods of making all these tests will be found in the books, but the writer has found them, as described, tedious and not very satisfactory. The methods now to be described are (in most cases) modifications of them, and we will first take adjustments (*a*), (*b*), and (*c*) together.

A point such as a first-floor window on one side of a wide street or square is chosen, so that, if a photograph be taken from it, we shall obtain a number of well-marked points (not less than twelve) none nearer to the instrument than about 100 feet, and giving images distributed all over the plate.

The points may be chosen with the aid of the ground-glass screen. They may well be arranged in three or four horizontal rows, of which the upper- and lower-most are near the top and bottom of the plate respectively; but they should not be in vertical rows. It is undesirable that the horizontal angle between any point and the next should be less than about one degree, or the *horizontal* distance between their images on the screen less than about 0·1 inch, whether the points be on the same level or not.

The relative directions of all these points are then fixed by careful theodolite observations of the horizontal angles between them. The angles should be read at least twice, preferably with reversed face (until agreement within 30″ is obtained), and the mean values plotted twice over on smooth paper, with a protractor reading to minutes, as shown (once only) on Fig. 118.

On a slip of tracing paper, a fine pencil line, VY, is ruled, and a point X marked on it to represent the vertical hair. The distances X1, X2, etc., are set off from X, left or right, as obtained from the negative, and the tracing paper is then moved about over the rays, until the position is found where each point, thus plotted on the tracing, most nearly falls on its own corresponding ray.

If every point is found to fall on its own ray with an error less than the probable errors of drawing and measurement, it may be taken that the vertical hair is sufficiently nearly vertical, and that there is no serious distortion.

The line VY is then pricked through on to the drawing; a perpendicular AX′ is drawn to it from A. Then AX′ will tell the true focal length.

If X′ and X coincide, the vertical hair is in the right place; if not, the distance X′X will tell how much it is displaced, and which way.

The distances should be set off again (*not* pricked through) on a separate slip, and the whole process repeated with the second set of rays. The agreement (or otherwise) of the results will show the kind of accuracy obtained.

If any discrepancies are found, and confirmed by the second test (*i.e.* if certain points will not coincide with their rays), these may be due to—

(1) the hair or plate not being truly vertical;
(2) local movements of the photographic film;
(3) local distortion of the image.

(1) To find out if the hair is vertical, there should be at least four (preferably five) points in a roughly horizontal row near the top of the negative, and a like number near the bottom (the middle rows may contain fewer points).

The test, as above, should now be made, using first the upper set of points only, then the lower set only, and remembering that each result must be confirmed on the second drawing.

If different values of XX′ are obtained and confirmed, this shows that the hair is not vertical, and the difference shows the amount of slope in a length equal to the vertical difference between the rows of points.

If we look at the film side of a negative right way up—that is, so that the sky is uppermost—points whose images are on our *left* on the negative will be to the *right* on the actual ground as we face the same area, and *vice versâ*.

Now, the rays (Fig. 118) mark the actual directions of the points as seen from A. Hence, to obtain coincidence, all distances which are to the *left* of

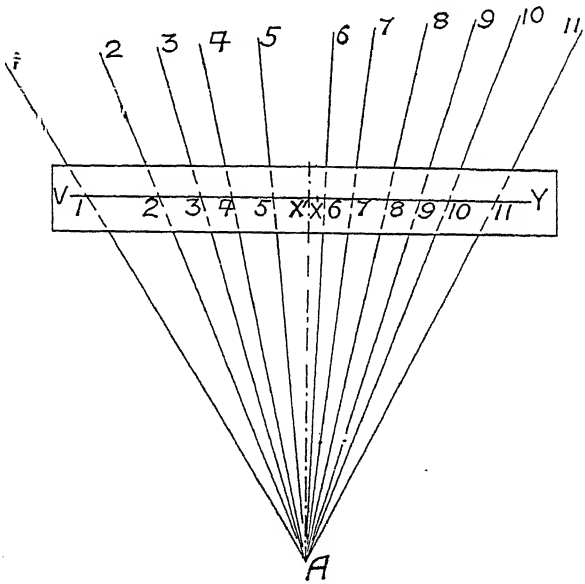


FIG. 118.

the negative in the above position must be taken to the *right* on the tracing slip, and so on.

Now, suppose in the above test for the five upper points, we get good agreement for those points, and we find X′X = 1·3 mm., X to the right of X′, as seen in Fig. 118; and for the lower points X′X = 0·9 mm., X again to the right of X′, and both results confirmed by the second drawing (or both representing the mean values if there is a *slight* difference between the drawings).

Then we infer that, looking at the film side of negative, right way up, the hair is 1·3 mm. too much to the *left* at the level of the upper row of points, and 0·9 mm. too much to the *left* at the lower level. We can rule a new line on the negative these distances to the *right* of the hair, and by remeasuring from this line and repeating the test with *all* the points we should obtain much better agreement.

In this case, a line should be ruled in this correct position on every negative, before any measurements are made from it, and a line must be ruled by trial and error, in the corresponding position on the ground glass screen to be used for setting to zero on the base line. If the hair is too much to the *left* when looking at the film side of negative, right way up, then it is also too far *left* as viewed on the actual camera from the back.

As a matter of fact, however, in the opinion of the writer, a slight displacement of the vertical hair (provided that it *is* pretty nearly vertical) is of little or no importance. For this reason it is very difficult, by any test, to obtain very close agreement in the values of X'X if the tests are independent. So that, almost certainly it will be found necessary to make two or three tests off each drawing and take the mean. And if we get nearly the same result for the upper and lower points, and not more than say 0·5 mm., we may treat the hair as correct throughout.

If the focal lengths obtained for the upper and lower points be different, the inference is that the plane of the plate is not vertical. If the focal length for the top points be the shorter, that part of the plate is inclined towards the lens.

(2) **Movement of Film.**—Local movement of the film can be detected by taking more than one photograph. Preferably at least three should be taken for the test, and a comparison of the measurements will show if any appreciable movement has taken place.

From a batch of five photographs taken at the same time, the writer found that there was local movement of the film, amounting in one case to as much as 0·6 mm.

In comparing photographs it is necessary to work from point to point, using the hair as an intermediary only, as there may have been some slight movement of the camera as a whole.

The practice of the writer is to take the point nearest to the hair as a standard, and to obtain the horizontal distance from that point to each of the others.

It will be obvious that any possible movement of the film should be eliminated by one or all of the above methods before testing for verticality of the hair.

Distortion.—When both movement of film and slope of hair have been allowed for as above, any measurable discrepancy between any one or more points and their corresponding rays in the test shown in Fig. 118 (or general disagreement), if confirmed by the second drawing, will indicate some kind of distortion of the image, and the instrument should be returned for a new lens.

Other Tests.—Separate tests for the different adjustments can readily be devised, but the writer thinks the combined test here described most suitable.

A similar test put into mathematical form and solved by the method of least squares is described in a paper on "Photographic Surveying" by the present writer in the *Transactions of the Irish Institution of Civil Engineers* for April, 1915, from which most of the above is taken by permission of the Council.

Scale of Angles.—Sometimes a scale of angles, left and right of the hair, is supplied by the makers.

The focal length can be found from this if desired. Thus if x be the measured distance from the hair to the point marked say 20° on the scale, and f be the focal length, then

$$f = x \cot 20^\circ$$

Of course, however, no careful surveyor would trust such a scale without testing it. The value of f should first be found by one of the methods described, and then checked by this formula. If the results do not agree, the scale of angles is incorrect, and should be discarded.

If they *do* agree, all horizontal measurements may be taken in degrees from this scale, and the rays plotted with a protractor if desired, but the method is not recommended.

Principal Line and Plane.—It has been pointed out that a slight lateral displacement of the vertical hair is of little or no importance, and it cannot be easily adjusted.

If the error be less than 0·5 mm. it can be ignored; or, if desired, the corrected position of the vertical hair can be ruled on the ground glass screen, and on each negative, by comparison with the existing hair.

If we look at the actual hair from the back of the camera, then the hair is actually too much to the *left* if we found it too much to the *left* on the negative when looking at the film side, sky line upwards.

The important things are that the hair should be vertical, that the focal length should be accurately known, and that the exact line which is to be taken as zero for measurements from the negative should be the same line which is set to zero on the base line AB. This is necessary in order that the true value of the angle BAZ (Fig. 115) may be known.

It is for this reason that we must rule the corrected position of the hair on the ground glass, unless the hair itself can be taken as correct.

When the exact line from which the measurements are to be taken is decided upon (whether it be the hair itself or on one side of it), we will call this line the *principal line;* and the plane which contains that line and the centre of the lens is the *principal plane.*

Adjustment of the Telescope.—If the telescope is to be used for setting to zero on the base line, or for measuring the angles between stations, it is therefore essential that the line of collimation of the telescope should be parallel to the principal plane, as above defined (or that the angle between them should be known). This can be readily tested by setting the telescope on any well-marked point, and then examining the image on the ground glass screen with a magnifying glass to see if the image of the same point lies on the principal line.

If not, we may proceed in various ways, viz.—

(*a*) Bring the principal line on to the image by the tangent screw, and then adjust the diaphragm screws of the telescope to bring the hairs back to the point.

(*b*) If it be desired not to alter the line of collimation of the telescope but the whole telescope is adjustable on top of the camera, then

bring the principal line to the image as before, but bring the cross-hairs of the telescope back to the point by moving the whole telescope.

(c) If it be desired not to alter the adjustments at all, read the horizontal circle when the telescope is on the point; then bring the principal line on to the image, and read again. The difference tells the angle between the line of collimation of the telescope and the principal plane. This may be called the "collimation index error," and must be applied as a correction to all observed angles, in plotting the position of the principal plane, which is shown at AZ in Fig. 116.

Thus if we obtain the following readings:—

Telescope on point . . .	217° 31′
Principal line on image . . .	218° 5′
Collimation index error . . .	−0° 34′

The error here is *minus*. Then suppose we set the *telescope* along the base line with the vernier clamped at zero, and then turn it until the same vernier reads R. This, of course, gives a clockwise angle R: and if we then take a photograph, the correct value to be used for the clockwise value of the angle BAZ (Fig. 116) is R − 34′.

If we set to zero on B with the telescope, and obtain a clockwise angle R to another station by setting the *telescope* on that station too, then, of course, the true value of the angle is R simply, as *both* readings are taken with the telescope. The instrument, so far as the new station is concerned, is then used simply as a theodolite, and its action is quite independent of the principal line.

Focus.—It is desirable to test the instrument also for focus. Points over 100 feet away should be more distinctly focussed than close points, with a large aperture. This can be tested by examining the image with a magnifying glass.

Stop.—It is desirable to work as far as possible with a fixed stop. As some pictures may have to be taken with a green (or orange) screen, or in poor light, it is not desirable that the stop should be too small.

But whatever stop is to be used on the survey it is desirable that the tests above described should be made with the same stop.

Screen.—With the green screen orthochromatic plates must be used, and it is necessary to make a set of tests to find the relative exposures with or without the screen.

It is desirable also that when the tests are being made some of the photographs should be taken with the screen and some without, to find out if it has any influence on the focal length.

Testing Horizontal Hair.—If a truly horizontal plane be drawn through the centre of the lens, this is called the "horizon plane." The plane of the photographic film is the "picture plane," and the line in which the horizon plane meets the picture plane is called the "horizon line."

When the instrument is carefully levelled, the horizontal hair should

coincide with the horizon line, and if the photographs are to be used for finding levels, even roughly, this should be tested.

For this purpose the difference in level between the station and, say, at least five or six points should be known, as well as the distances to the points.

It is a good plan (though not essential) to have these points some below and some above the hair, but not too near it, and they should vary in position from the extreme left to the extreme right. The points chosen may be the same as those already used for the previous test, if a sufficient number of them are conveniently situated, so that their distances and differences of level from the station can be found by any ordinary surveying method.

Then if L = level of station, h = height of camera, we have $L + h$ = level of horizon plane.

Now let L_1, L_2, etc., be the levels of the points.

Therefore the height of No. 1 point above the camera is $L_1 - (L + h)$; we will call this H_1.

Therefore $H_1 = L_1 - (L + h)$; $H_2 = L_2 - (L + h)$, and so on, where all dimensions are supposed to be in *feet*.

Now let d_1, d_2, etc., be the distances from the station to the points, also in feet, and let y_1, y_2, etc., be the measured distances on the negative by which the images of the points are above the horizontal hair. These are to be reckoned *minus* if the images are *below* the hair (looking at the negative as before), and may be measured in millimetres or any convenient units. Let f be the focal length (found as before described) and x_1, x_2, etc., the horizontal distances of the images left or right of the principal line.

These values must all be in the same units as y_1, y_2, etc., and so must the value of f.

Then, as already stated, we should have the following relation :—

$$H_1 = y_1 \times \frac{d_1}{\sqrt{f^2 + x_1^2}}, \text{ and so on}$$

Hence
$$y_1 = \frac{H_1 \times \sqrt{f^2 + x_1^2}}{d_1}$$

$$y_2 = \frac{H_2 \times \sqrt{f^2 + x_2^2}}{d_2}, \text{ and so on}$$

The various values of y are calculated by this formula, and then *subtracted from* the measured values. We will call the results E_1, E_2, etc.

It is to be understood that the calculated, as well as the measured, values of y may be *plus or minus*. In subtracting, we always subtract the calculated value *from* the measured, and perform the subtraction algebraically, so that E_1, etc., may also be plus or minus.

Now, suppose E_1 and E_2 are the results for two points, both near the extreme right (or left) of the picture, looking at the film side of negative, right way up.

Then E_1 and E_2 should have the same sign (unless they are smaller than the probable error of measurement), and they should not differ by an amount greater than the combined probable error of two measurements.

Otherwise there is either some distortion of the image due to a faulty lens, or the film has stretched. This last can be tested by a second picture.

If E_1 and E_2 nearly agree, the mean will tell how much the hair is too high or too low at that end (right or left as the case may be). It is *too high* if E_1 and E_2, found as above, are *minus*, and *too low* if they are *plus*.

But it must be carefully remembered that this means too high or too low *on the negative, when looked at right way up.*

In the camera the plate is, so to speak, upside down. Hence the hair *in the actual camera* is *too high* if E_1 and E_2 are *plus*, and *vice versâ*.

The other end of the hair should be tested in the same way, and checked by one or two points in the middle.

Any error can be corrected either by moving the hair; by ruling a new line in the proper position on the negative after taking a print; or by measuring from the hair and correcting the measurements.

If E be the error found as above at any part of the hair, and y the measured value, then the corrected value to be used in the formula on p. 308 is $y - E$, due regard being paid to sign.

Centre of Instrument.—It will be clear from what has been said that, properly, the vertical axis of the instrument and the station on the field should be vertically under the centre of the lens, and not under the centre of the camera.

There are some mechanical difficulties to be overcome in so constructing the instruments, and for distant points the error introduced by having the centre of the lens displaced two or three inches from the true station is not very important. This amount would not, in fact, be plottable to scale on the paper. Still, it is a point to be borne in mind. In the tests described (especially as the points may not be more than 100 feet away) the centre of the instrument used for measuring the angles between the points should be centred as nearly as possible over the same point on the ground which was covered by the *centre of the lens* when taking the test photographs.

Thus if the angles are measured with the photo-theodolite itself, the latter must be moved forward, after taking the photos (by an amount equal to the distance between the centre of the camera and the centre of the lens) before the angles are measured.

If new stations are fixed by measuring the horizontal angles to them directly with the telescope, this source of error is thereby eliminated, so far as stations are concerned, with an instrument like the Bridges-Lee.

Enlargements.—If enlarged photographs are to be used for plotting, no stated ratio of enlargement should be trusted.

A test photograph or two should be taken, as above described, and enlarged at the same time and in the same way as the rest, and all measurements used for the test should be taken directly from the enlarged photographs.

In this way the focal length, etc., to be used in plotting are found directly, and any distortion in enlargement is detected.

Barometric and Hypsometric Determinations of Heights above Sea-level.

The ever-decreasing pressure of the atmosphere as increased heights above sea-level are attained, suggests an obvious method

of determining differences of altitude by means of barometric observations.

With whatever instrument the air pressure or density at various altitudes has been determined, whether with a mercurial barometer, an aneroid, or by the temperature at which water boils with the hypsometer, the same calculations have to be made in order to work out the relative heights at which the various observations were taken.

Excepting when used as described above for sketch surveys, or for determining differences of level at moderate intervals and within limited intervals of time, more or less synchronous observations of the air's density must be made at the stations whose difference in altitude it is desired to determine. This is obviously necessary, since the density and pressure of the air are materially affected by changes in temperature, and such changes will generally take place if observations are made at considerable differences in altitude above sea-level, or at several hours' interval of time.

Observations should be taken in as settled weather as possible, and if near a fixed meteorological station, at the same hours of the day at which observations are taken at such a station. As many observations as possible should be taken for each determination. Though it is impossible, in the present state of physical science, to effect any high refinement in the formula for computing barometric heights, still, fair results can be attained if the limits of error due to the instrument and methods of observation are carefully weighed, and attention paid to the hour of day (for the diurnal wave variations) and the month of the year (for monthly mean variations), as also to the degree of unsettledness of the weather at the time the observations were taken, and effect given to the above in the calculation of the results.

From inattention to these simple considerations many important heights as published are really most erroneous, and require revision.

The Mercurial Mountain Barometer.—This portable barometer is constructed with a thumb-screw at its lower extremity, by turning which the mercury in the reservoir is confined within a limited space for transport purposes. In such cases this end should be carried uppermost.

The most accurate results possible are to be obtained with this barometer, but it requires more careful handling than the aneroid. Observations must be corrected for the readings of both the "attached" and "detached" (air) thermometers.

The Aneroid Barometer.—This well-known and useful instrument is made of various sizes, and is compensated for internal

effects of changes of temperature, but from this fact it does not follow that the calculations must not include corrections for the air temperature. This instrument must be frequently compared with a mercurial barometer, and even then none can lay any claim to the exactness of the latter. The mechanism is liable to get out of order, and they are slow in "taking up" the effects of changes of pressure.

Its portability and handiness are a great recommendation, no doubt, and in many cases (especially if several be used as a check on each other) good results may be obtained.

The Hypsometer.—The hypsometer is an instrument for determining the temperature of the boiling-point of water, from which barometric pressures can be directly deduced and used as above indicated to calculate heights above sea-level. It consists of a cylindrical reservoir with a telescopic funnel, from the top of which when drawn out and in use the steam escapes through a small hole, and through which the stem of a thermometer (whose bulb is immersed in the water in the reservoir) protrudes, and can be read. The reservoir is heated by a spirit-lamp.

Formulæ for Calculating Heights.—Various formulæ have been devised and tables calculated for computing the difference in height between two stations by means of the observed heights of the barometer, of which the following are examples.

(1) By Laplace—

$$\text{Difference in height in feet} = 60{,}345{\cdot}51 \times \{1 + 0{\cdot}00111111(\tau + \tau' - 64^\circ)\} \times \log\left\{\frac{\beta}{\beta'} \times \frac{1}{1 + 0{\cdot}0001(t - t')}\right\} \times (1 + 0{\cdot}002695 \cos 2\phi)$$

where β, β' = barometric readings,
t, t' = readings of "attached" thermometers,
τ, τ' = readings of air temperatures, or "detached" thermometers,
ϕ = latitude (mean of stations).

The *unaccentuated* figures are the readings at the *lower* of the two stations.

(2) By Poisson—

$$\text{H in yards} = \text{A}\left[\log \beta - \log\left\{\beta' \times \left(1 + \frac{t - t'}{9990}\right)\right\}\right]$$

$$\text{where A} = \frac{20053{\cdot}95}{1 - 0{\cdot}002588 \cos \phi}\left(1 + \frac{\tau + \tau' - 64^\circ}{900}\right)$$

(3) Given by Colonel Mackesy in his "Tables of Barometric Heights" for use with the compensated aneroid barometer.

$$\left.\begin{array}{c}\text{Difference in height in feet}\\ \text{between two stations}\end{array}\right\} = 67{\cdot}05(836 + \tau + \tau') \times (\log \beta - \log \beta')$$

This formula can be used with the mercurial barometer if the readings are first corrected for capillarity and reduced to 32° F.

(4) To connect observations of the boiling-point as indicated by the hypsometer, with corresponding barometric readings.

$$h = 29{\cdot}92 - 0{\cdot}59t$$

where h = corresponding height of mercurial column in inches,
t° = degrees F., by which the boiling-point, as read on the hypsometer, is less than 212° F.

The air temperature must be taken at each observation for use in the formula selected for calculating altitudes by the observed (in this case calculated) heights of the barometric column.

Note.—In both formulæ the factor involving τ, τ', the readings of the air thermometer or "detached" thermometer at upper and lower stations, corrects for the mean density of the intervening air stratum, which is *assumed* to be the arithmetical mean of the upper and lower air-temperatures. This assumption is not always correct. Hence barometric levels are always uncertain.

The factor containing t and t' corrects the barometric reading for the effect of temperature on the mercury and scale, and is therefore unnecessary with the aneroid and with the hypsometer.

(5) An approximate formula given in "Military Sketching made Easy," by Major-General Hutchinson, is

$$H = 52{,}500 \frac{D}{S}\left\{1 - \frac{1}{3}\left(\frac{D}{S}\right)^2\right\}$$

where H = difference of level in feet, D = difference in barometric heights, and S = sum, both in inches.

Investigation of Formula for Levelling with the Barometer

The method is based on the fact that the pressure of the air decreases as we ascend; firstly, because there is a less actual depth of air above the higher level, and also because the density of the air decreases with the elevation.

Suppose A and B (Fig. 119) to represent two stations at different levels. Let BM = X be the difference of level.

Let R be any point in AB whose height above AM = x, and let p = pressure of air per square foot at R. Let Q be a point a little higher, so

that the difference of level from R to Q $= dx$, and let dp be the corresponding change in pressure, which will be negative. This loss of pressure is due to, and equal to, the weight of a column of air 1 sq. ft. in the base, and whose height is equal to the difference of level dx.

This latter being small, the density of the air may be regarded as constant from R to Q. Denote the weight of 1 cub. ft. of the air by δ.

$$\text{Then volume of column} = 1 \times dx = dx \text{ cub. ft.}$$
$$\therefore \text{ weight of column of air} = \delta \times dx \text{ lbs.}$$
$$\text{But this equals change in pressure} = -dp$$
$$\therefore \delta \times dx = -dp \quad \ldots \ldots \quad (1)$$

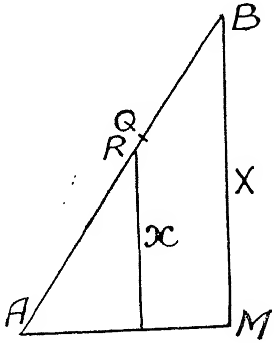


FIG. 119.

We must now find δ, the weight of 1 cub. ft. of air at R. Let w = weight of 1 cub. ft. of air at a temperature of 32° F., and under a standard pressure P, and in the latitude of 45°.

Then, by the laws of the expansion of gases, if δ be the weight of a cub. ft. at any other temperature $\tau°$ F. and pressure p in the same latitude,

$$\frac{\delta}{w} = \frac{p}{\mathrm{P}} \cdot \left(\frac{493}{493 + \tau - 32}\right)$$
$$\therefore \delta = \frac{w}{\mathrm{P}}\left(\frac{493}{493 + \tau - 32}\right) \cdot p \quad \ldots \ldots \quad (2)$$

If the latitude change to ϕ, the force of gravity changes also to $\dfrac{1}{1 + 0{\cdot}002695 \cos 2\phi}$ of its value at 45° latitude.

Hence the weight of a cub. ft. of air at latitude ϕ, pressure p and temperature $\tau°$ F. becomes—

$$\delta = \frac{w}{\mathrm{P}}\left(\frac{493}{493 + \tau - 32}\right) \cdot p\left(\frac{1}{1 + 0{\cdot}002695 \cos 2\phi}\right) \quad \ldots \quad (3)$$

Substitute this in (1) and we get

$$\frac{w}{\mathrm{P}}\left(\frac{493}{493 + \tau - 32}\right)\left(\frac{1}{1 + 0{\cdot}002695 \cos 2\phi}\right) p \times dx = -dp$$

If we suppose τ constant from A to B we may write this—

$$Cpdx = -dp$$

where C is constant and equal to

$$\frac{w}{P}\left(\frac{493}{493+\tau-32}\right)\left(\frac{1}{1+0{\cdot}002695\cos 2\phi}\right)$$

or

$$Cdx = -\frac{dp}{p}$$

Integrating,

$$CX = -\int_{P_1}^{P_2}\frac{dp}{p}$$

$$= \log_e\left(\frac{P_1}{P_2}\right)$$

where P_1 and P_2 are the pressures at A and B respectively.

$$\text{Hence } CX = \log_e\left(\frac{P_1}{P_2}\right) \text{ if } \tau \text{ be constant} \quad . \quad . \quad . \quad . \quad (4)$$

In practice τ is not constant, and hence a mean value is taken for it, viz. $\frac{\tau + \tau_1}{2}$, where τ and τ_1 are the air temperatures (as read by the detached thermometers) at A and B.

If the temperatures of the mercury, as read on the attached thermometers, are different at the two stations, a correction must be introduced to allow for the expansion of the mercury and scale, whence we derive the corrective factor $\frac{1}{1+0{\cdot}0001(t-t')}$ on p. 320.

If we take P in lbs. per sq. ft. and w in lbs., and substitute proper values for them, as obtained by experiment, we shall arrive at a result in near agreement with the above formula.

The unavoidable error in finding the mean temperature of the air generally far exceeds that due to neglecting the corrections for latitude and for attached thermometers, and renders levelling by the barometer always uncertain.

It is best to take a number of readings of the air temperature at intermediate points, and so get a nearer approximation to the true mean.

If it is desired to correct for the expansion of the mercury and the scales, it is perhaps more convenient, as well as more exact, to reduce each barometric reading separately to freezing-point.

The Hypsometer

It has been found, experimentally, that an increase of pressure of about 26·8 mms. of mercury causes a difference of 1° C. in the boiling-point of water.

Hence, $\frac{5}{9} \times 26{\cdot}8$ mms., or 0·586 inch of mercury will cause a difference of 1° F. if the change in boiling-point be assumed proportional to that in pressure.

This is not quite true, and it is better to obtain from tables the pressures corresponding to the boiling-points; but, on the above assumption, remembering that at 29·92 ins. of mercury water boils at 212° F., we obtain the result that, if the boiling-point differs from 212° by t, then the pressure will be $29{\cdot}92 \pm t \times 0{\cdot}586$, according as the boiling-point is higher or lower than 212°.

Another formula, given by Cotterill, and more exact, is

$$\log p = 5 \, . \frac{t - 212}{t + 367} + \log 14{\cdot}7$$

where p is the pressure in lbs. per sq. in. and t is the Fahrenheit temperature.

The following table from Cotterill's treatise on the "Steam Engine," gives the pressures corresponding to a few boiling-points.

Boiling-point (°Fahrenheit)	208	209	210	211	212	213	214	215	216
Pressure (lbs. per sq. in.)	13·57	13·84	14·12	14·41	14·70	14·99	15·29	15·60	15·91

The Steward Hypsometric Aneroid

It is claimed for this instrument that by its use the measurement of altitudes is simplified.

Description.—The barometric portion of the instrument does not differ from those in general use.

The altitude scale forms a complete circle of equal divisions, and the zero is adjustable. Ascents and descents are placed to the left and right of zero respectively. The altitude scale is automatically locked and cannot shift during transit.

It is claimed for this instrument that the difference of level between stations can be read directly without calculation. With an altitude scale extending to 10,000 feet, differences of 5 feet or less can be read without the use of a vernier, and there is no error of parallax.

A small swing thermometer for ascertaining the air temperature, to make the correction for difference in the weight and volume of the atmosphere, is attached.

Particulars can be obtained from the makers, Mr. J. H. Steward, Strand, London.

Parliamentary Surveying

In this country parliamentary surveying is often an important part of a surveyor's work. It differs from what has already been

described in this book, chiefly in the fact that it is usual to start with the Ordnance Survey maps already in existence, and merely to check these on the field, showing additions, etc., within the limits of deviation.

The requirements are fully given in the "Standing orders," etc., to be obtained from the Government publishers. The methods to be used will necessarily depend upon the nature of the ground, but perhaps the tacheometer and plane-table are among the most useful instruments for this work. The parliamentary surveyor must remember that the survey is not intended for a *detailed* estimate of quantities or anything of that kind, and that speed is essential with such a survey. At the same time, some care in setting out centre lines and so forth may save much trouble afterwards. He must also remember that in conducting a parliamentary survey he is really, in general, trespassing, and must exercise great care with regard to damage of any kind.

CHAPTER X

MAP PROJECTIONS

Preliminary Remarks.—If all boundary and other lines on the earth's surface, as well as all isolated points such as mountain peaks, were projected vertically on to the surface of the spheroid passing everywhere through mean sea-level, a map should give a true representation of all such points and lines.

It is, perhaps, most convenient to suppose this spheroid to shrink uniformly until its size becomes such that the country to be mapped covers an area, on the reduced spheroid, no greater than that of the paper on which the map is to be drawn.

Then if the shape were such that the surface was developable, it would only be necessary to place a piece of tracing-paper on the reduced model, trace the lines, etc., to be mapped, and then flatten out the paper. As, however, a spheroidal surface cannot be developed in this way, tracing-paper could not be made to lie flat on it without getting into folds and creases.

The consequence is that *no* map of any large area, drawn on one flat sheet of paper, *can* be correct in all particulars.

The title "map projection" is used to cover the study of the different methods adopted for overcoming the difficulties as far as possible.

It is not possible here to do more than touch upon some of these methods, and we shall suppose that the lines to be represented are meridians of longitude and parallels of latitude at stated intervals apart, as when these are shown the rest of the survey can clearly be interpolated.

The Perfect Map.—In the perfect map the *scale* should be everywhere the same, both along parallels and along meridians; meridians should be represented by straight lines and parallels by curves (as they *are* on the actual earth), and meridians and parallels should intersect always at right angles.

Simple Conical Projection.—In Fig. 120, let PQ be the polar axis, Q the centre of the earth. Let AA_1 be some chosen parallel of latitude at about the centre of the area.

Then a cone is described touching the reduced spheroid along

AA_1, the apex being clearly at P. With P as centre and PA_1 as radius, describe a circular arc A_1A_2.

Now make the length of the arc A_1A_2 equal to the circumference of the parallel AA_1, and join PA_2. PA_1A_2 represents the development of the cone.

Now divide A_1A_2 to represent the required longitude intervals, as shown at 1, 2, etc., and join to P. These lines represent the meridians.

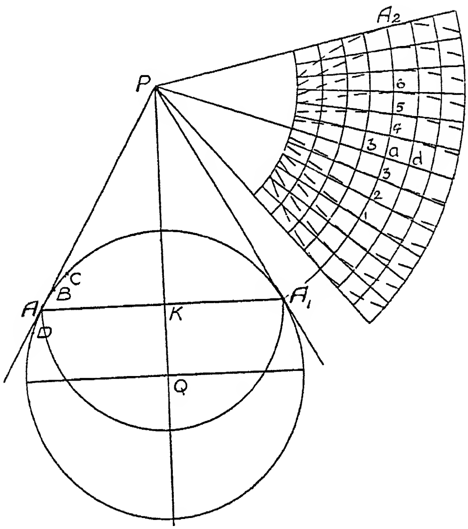


FIG. 120.

Now along the central meridian Pa set off ab, ad, etc., to represent the lengths of the arcs of meridian AB, AD, etc., corresponding to the desired latitude intervals, and draw arcs through these points with P as centre. These will give the parallels.

In the actual figure, only one-third of the parallels is shown.

Here meridians are straight, parallels curved. They meet everywhere at right angles, and the scale along any meridian anywhere is the same as along the central parallel. But all other parallels are *too long*; that is, longer than the actual lengths on the spheroid would be if drawn to the same scale as the meridian arcs.

A formula has been given (p. 227) for finding the radius AK of any parallel. Now it is clear that the angle PAK (Fig. 120) is the complement of the latitude, λ, of the standard parallel, and the radius, PA_1, of this parallel in the projection is therefore AK $\times$ cosec λ, where AK is the actual radius of the parallel on the spheroid.

The radii for other parallels are found by adding (or subtracting) the lengths of the arcs of meridian between the parallels.

The resulting radii being usually too great for direct construction by compasses, it is best to calculate x and y co-ordinates, taking Pa (Fig. 121) as the axis of y, and the tangent at a as the axis of x.

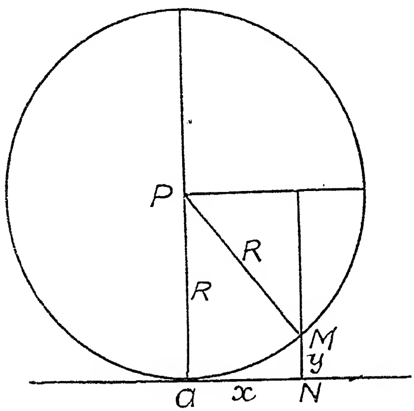


Fig. 121.

In Fig. 121, if Pa = PM = R; aN = x; NM = y, it is clear that $R^2 = x^2 + (R - y)^2$, whence $x^2 = 2Ry - y^2$, so that x can easily be evaluated for different values of y.

This is called the simple conical projection with *one standard parallel*, and is clearly suitable for an area having a large range of *longitude*, but not much range in *latitude*, as the scale rapidly increases away from the central parallel.

Bonne's Projection.—To do away with this objection, in Bonne's projection the parallels are drawn, and the central meridian divided, exactly as in the simple conical, but instead of drawing the meridians as straight lines through the points 1, 2, etc., *each* parallel is made its true length, and divided up exactly in the same way as the standard parallel. The points of division are joined to draw the meridians, which are therefore curved as shown by the broken lines in Fig. 120.

The scale is correct along all parallels, but meridians are slightly too long and do not cut the parallels at right angles, the error increasing as we depart from the central meridian. The projection is therefore useful for a large range in *latitude*, without much range in *longitude*.

Two Standard Parallels.—By arranging that *two* parallels, one near the top and the other near the bottom of the area, shall be

made their correct lengths, the range of latitude shown on a simple conical projection can be increased. If this projection be shown in Fig. 122, the standard parallels AA_1, CC_1 are to be their correct lengths, and ac is to give the correct distance between them along the meridian. Now all these lengths are known, according to the accepted figure of the earth. And we have

$$aP : ac :: AA_1 : AA_1 - CC_1$$

This gives the distance, along ac produced, to the centre P with which the parallels are to be described. The central meridian is divided as before. *Either* standard parallel is divided, and the meridians ruled in as straight lines radiating to P.

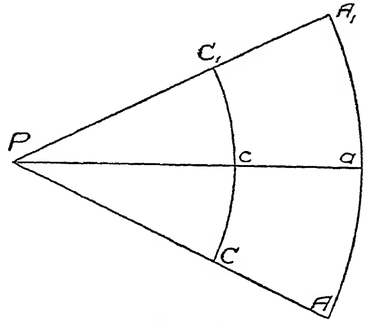


FIG. 122.

This is called the simple conical projection with two standard parallels.

The scale along the parallels is *too small* between the standards and *too great* outside the standards. Mr. Hinks recommends that the standards should be about one-seventh of the whole range in latitude from the top and bottom for a fair general result.

Conical Equal Area and Orthomorphic.—It is clear that, in the last projection, areas *between* the standard parallels will be *too small* (as the parallels are too short), while outside they will be too great. This can be overcome by *increasing* the distances along the meridians in the central part (and *decreasing* them outside) to the same extent as the parallels are too short or too long. This leads to the idea of a *conical equal area* projection, but, though *areas* will be correct everywhere, it is clear that the scale along meridians and parallels will be nowhere the same (and neither will be correct) except in the immediate vicinity of the standard parallels.

On the other hand, we may wish to arrange that, at any point of the map, the scale shall be the same along meridians and parallels. In this case we must *decrease* distances along the meridian in the central zone (where the parallels are *too short*), and *increase* them *outside*.

A projection in which this property is secured (together with perpendicular intersection of the meridians and parallels) is said to be *orthomorphic*, and leads to a correct *shape* for any *small* area. But *areas* are incorrect.

Thus, in this case it is clear that areas *between* the standards will be *too small,* and those *outside too great,* even more than in the simple conical. The scale will also be too small in all directions between the standards, and too great outside.

Polyconic Projections.—Suppose that, instead of describing all the parallels with the same centre P (Fig. 123), a new cone is imagined for each parallel to be drawn, touching the spheroid along that parallel. Now let the central meridian be divided up at a, b, c as before. Set off ap = AP, bq = BQ, cr = CR, and so on, and describe the parallels with p, q, r, as centres. Make each

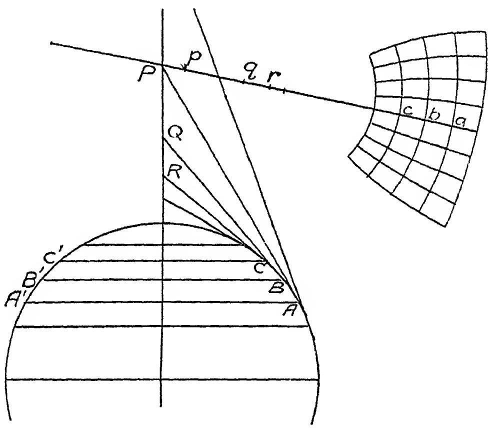


FIG. 123.

parallel its proper length, and divide up, drawing the meridians by joining the points of division, as in Bonne's. This is called the polyconic projection. It is evident that all meridians except the central one are too long.

International Map.—The projection adopted for the new international map of the world on the scale of 1 : 1,000,000 is a modified polyconic. The sheets may be regarded as bounded by meridians and parallels. The latter are constructed as described above, except that the distances along the central meridian are made slightly *too short,* so that the meridian scale shall be correct at *one-third the range of the sheet from either end,* instead of in the middle only. The *top* and *bottom parallels* on each sheet are truly divided, and the meridians are drawn as straight lines through these points of division.

Each sheet then fits correctly with each of the four adjacent sheets.

Sanson-Flamsteed.—The Sanson-Flamsteed or sinusoidal projection is an equal area projection of the whole world. The central meridian AB is divided correctly, the parallels being drawn as straight lines through the points of division. Each parallel is made its true length and correctly divided. It is therefore a special case of Bonne's, with the equator as standard parallel.

Mercator's Projection.—In the Mercator, the parallels are drawn as straight lines *all of equal length*, while, in order to make the projection orthomorphic, the distances between the parallels

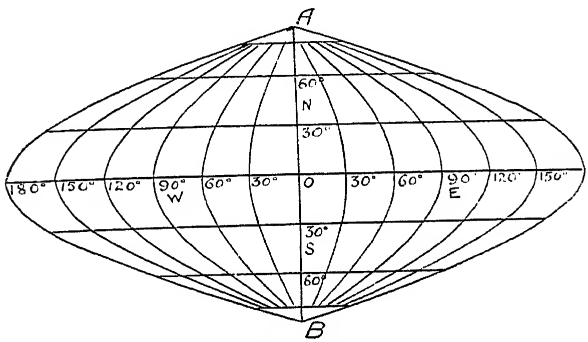


FIG. 124.

are increased as we move away from the equator, in the same ratio as the parallels are too long.

Zenithal Projections.—In the class of projections known as zenithal, a point at about the centre of the area is chosen as the centre of the map, and all straight lines radiating from this are shown as radiating straight lines making the proper angles with one another on the map.

In the zenithal equidistant, distances along these lines are also made correct. Thus if the pole were the centre, the radiating straight lines would represent meridians, and, these being properly divided, the parallels are described as circles through the points of division with the pole as centre.

The scale of the parallels gets rapidly too great away from the centre.

In the zenithal orthomorphic (stereographic) the scale along the meridians is increased in the same ratio as the scale of the parallel.

In the zenithal equal area, the scale along meridians is *decreased* in the same ratio, so as to keep areas correct.

There are many other projections in use, which cannot be here described.

For a fuller account of these, the reader is referred to the books on map projections, such as that by Mr. A. R. Hinks.

Problems which have generally to be solved for Mapping Purposes.—Having indicated the various methods which may be employed when mapping surveys, it is now desirable to state and explain the problems which have most generally to be solved in such cases. They are as follows :—

(1) Given the "lat." and "long." of one station, together with the "distance" and observed "azimuth" to another, to compute the "*lat.*" *and "long." of the latter*, together with *the reverse "azimuth" to the first.*

Let A (Fig. 125) be the station whose latitude and longitude are known, and suppose P is the pole, and that the distance AB, in feet, is given, as well as the azimuth PAB.

Let AQ be the vertical at A, meeting the polar axis at Q, and let a sphere be described with centre Q, to pass through A, as shown by the dotted line.

Let B_1, C be the points in which this sphere is intersected by QB and QP produced. Then in the spherical triangle AB_1C,

$$AC = AQP = 90° - \text{latitude of A}$$
$$CAB_1 = \text{angle between planes PAQ, BAQ}$$
$$= \text{azimuth of B from A} = \text{A (say), and both these parts are known.}$$

Now, as A and B are necessarily not very far apart, it may be taken without practical error that B and B_1 are coincident (remembering that all distances have been reduced to mean sea-level).

Hence to find the side AB_1, we reduce the distance AB to angle, using for radius AQ, which is the length of the normal at A terminated by the polar axis. A table to facilitate this reduction is given on p. 350.

We can therefore solve the triangle ACB_1 to find (*a*) the angle ACB_1. This gives the angle between the planes APQ and PQB, and therefore the *true difference of longitude* between A and B. (*b*) The angle AB_1C, which is the angle between the planes ABQ and QBP.

Now if BS be the normal at B to the spheroidal earth, the azimuth of A from B (west of north) is the angle between the planes ABS and SBP.

Of these the plane SBP coincides with QBP, and ABS would coincide with ABQ if BS and AQ were to meet in one point, say R.

Actually, as we have previously stated, we may consider that for any observable distance AB, the normals at A and B meet at a point R representing the centre of the sphere which most nearly coincides with the surface of the true earth at the middle of AB.

Hence we may take it, without practical error, that the calculated angle AB_1C in the above triangle gives the *azimuth* of A from B.

(*c*) The side CB_1, which gives the measure of the angle BQP.

Now, the co-latitude of B is given by the angle BSP, which may differ from BQP by an appreciable amount, even in moderate distances.

Now suppose D on the sphere be a point such that the angle PQD = PQB, or the side PB_1 found as above. Then if DQ be joined to meet the spheroid at D_1, it may be taken that D coincides with D_1.

Now the latitude of D is the same as that of B; and the length of arc

$$AD = (AQC - BQP) \times AQ$$

Again, the true difference of latitude from A to D is the angle which AD subtends at the centre of curvature of the meridian AD.

Hence if ρ be the radius of curvature along the meridian at A, we have true difference of latitude A to B = arc AD $\div \rho = (\text{AC} - \text{B}_1\text{C}) \times \dfrac{\text{AQ}}{\rho}$.

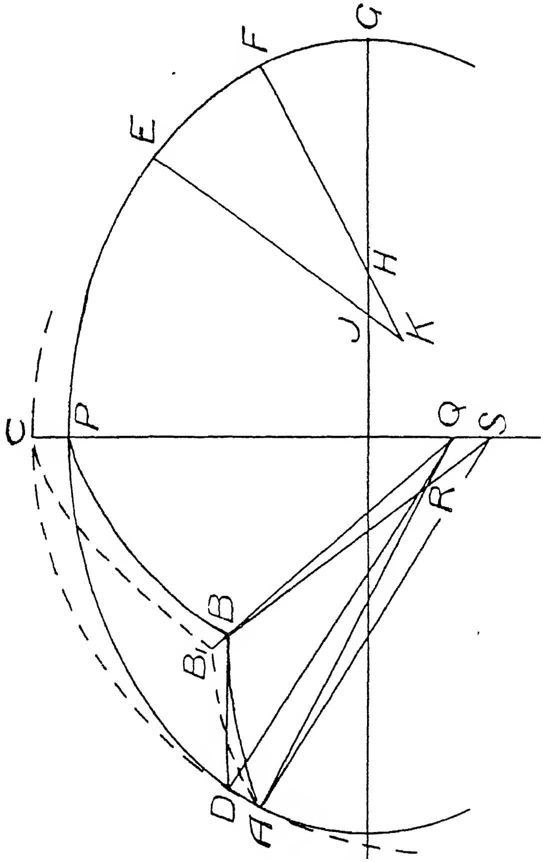


Fig. 125.

Thus if B_1C as calculated by the spherical triangle $= a$; AC = given co-latitude of A $= b$; AQ $= \nu =$ length of normal at A terminated by minor axis; $\rho =$ radius of curvature along meridian; then

$$\text{true difference of latitude} = (b - a) \times \frac{\nu}{\rho}$$

It is clear that AQ, or ν, is the radius of curvature at right angles to the meridian, and formulæ for ρ and ν are given on p. 230.

Table II. (p. 351) gives the value and logarithm of $\frac{\nu}{\rho}$.

At station A, given,

$$\left.\begin{array}{lll} \text{Latitude} & = 12^\circ \text{ N.} & = 90^\circ - b \\ \text{Longitude} & = 3^\circ \text{ E.} & \\ \text{Distance AB} & = 15 \text{ miles} = c & \end{array}\right\} \text{(Fig. 125)}$$

Azimuth of B from A = 135° (measured from S. by W.).

Required the lat. and long. of B, and azimuth of A from B.
It is first necessary to reduce c to "distance in arc."
This is done by the use of Table I. with latitude 12° and distance in feet (79,200), thus:—

$$\begin{aligned} 70{,}000 \text{ ft.} &= 689''\cdot87 \text{ at lat. } 12^\circ \\ 9000 \quad ,, &= 88''\cdot697 \\ 200 \quad ,, &= 1''\cdot971 \\ c &= 780''\cdot54 = 13'\ 0''\cdot54 \end{aligned}$$

For the formulæ for solution, see p. 34.

$$\frac{b+c}{2} = 39^\circ\ 6'\ 30''\cdot27 \qquad\qquad A = 45^\circ$$

$$\frac{b-c}{2} = 38^\circ\ 53'\ 29''\cdot73 \qquad\qquad \frac{A}{2} = 22^\circ\ 30'$$

$$\begin{array}{rl|rl} \text{L}\cos\frac{b-c}{2} = & 9\cdot8911668 & \text{L}\sin\frac{b-c}{2} = & 9\cdot7978551 \\ \text{L}\cos\frac{b+c}{2} = & 9\cdot8898359 & \text{L}\sin\frac{b+c}{2} = & 9\cdot7998846 \\ \hline & 0\cdot0013309 & & \bar{1}\cdot9979705 \\ \text{L}\cot\frac{A}{2} = & 10\cdot3827757 & & 10\cdot3827757 \\ \hline \text{L}\tan\frac{B+C}{2} = & 10\cdot3841066 & \text{L}\tan\frac{B-C}{2} = & 10\cdot3807462 \end{array}$$

$$\frac{B-C}{2} = 67^\circ\ 33'\ 43''\cdot24$$

$$\frac{B-C}{} = 67^\circ\ 24'\ 18''\cdot65$$

Sum = B = 134° 58′ 1″·89 E. of N.
Diff. = C = 9′ 24″·59 W. of A.

∴ Az. of A from B = 314° 58′ 1″·89 W. of S.
and long. of B = 2° 50′ 35″·41 E.

$$\begin{aligned} \text{L}\sin A &= 9\cdot8494850 \\ \text{L}\sin b &= 9\cdot9904044 \\ & 19\cdot8398894 \\ \text{L}\sin B &= 9\cdot8497335 \\ \text{L}\sin a &= 9\cdot9901559 \\ \therefore a &= 77^\circ\ 50'\ 48''\cdot4 \end{aligned}$$

and approx. lat. of B = 12° 9′ 11″·6

Correction of Latitude

$$(b - a) = 9' \, 11''\cdot 6 = 551''\cdot 6$$

$$\begin{aligned} \log 551\cdot 6 &= 2\cdot 7416243 \\ \text{(for lat. } 12^\circ \, 4\tfrac{1}{2}'\text{) } \log \text{Tab. II.} &= \underline{0\cdot 0028278} \\ &\ 2\cdot 7444521 \\ &= \log 555\cdot 23 \\ \therefore \text{ true diff. of lat.} &= 9' \, 15''\cdot 23 \\ \therefore \text{ true lat. of B} &= 12^\circ \, 9' \, 15''\cdot 23 \text{ N.} \end{aligned}$$

(2) Given the latitudes and longitudes of two stations, to compute the distance between them and their mutual azimuths.

Problem (2).—In this case we are given the "latitudes" and "longitudes" of A and B, *i.e.* we know a, b, and C.

We require the "distance" c, and the "azimuths" A and B.

The formulæ will be similar to those used above—

$$\tan \tfrac{1}{2}(\mathrm{A} + \mathrm{B}) = \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} \cot \frac{\mathrm{C}}{2} \quad \ldots \ldots \quad (d)$$

$$\tan \tfrac{1}{2}(\mathrm{A} - \mathrm{B}) = \frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2}(a + b)} \cot \frac{\mathrm{C}}{2} \quad \ldots \ldots \quad (e)$$

$$\frac{\sin c}{\sin b} = \frac{\sin \mathrm{C}}{\sin \mathrm{B}} \quad \ldots \ldots \ldots \ldots \quad (f)$$

Before using the *spherical* formulæ the reverse of the operation for correcting the "latitude"—explained above for Problem (1)—must be gone through.

i.e. log *spherical* diff. of lat. = log $(b - a)$ — log from Table II.

Taking b (*i.e.* 90° — lat. of A) as correct, we compute from this formula the difference of latitude as measured on a sphere with centre Q (Fig. 125). This is added to the given latitude of A (assuming that A is the point of lower latitude). The result gives the spherical latitude of B, and its complement is the value of a to be used in the spherical formulæ above.

When c is found, this gives the angle AQB, and we reduce back to feet by Table I., or using the radius of curvature at right angles to the meridian (p. 230).

If desired, both a and b may be corrected to reduce everything to the mean latitude of A and B. As we have seen, if a_1 and b_1 be the true latitudes, and a and b those used in the formulæ, then $(a - b) = (a_1 - b_1) \times \frac{1}{\mathrm{K}}$, where K is the natural number from Table II. Therefore if $a = a_1$, the *correction* to be applied to the true difference of latitude $(a_1 - b_1)$ is

$$(a_1 - b_1)\left(1 - \frac{1}{\mathrm{K}}\right) = (a_1 - b_1)\left(\frac{\mathrm{K} - 1}{\mathrm{K}}\right)$$

We may apply this equally to the latitudes of A and B.

$$\begin{aligned} a &= 90^\circ - \text{latitude of B} + \mathrm{P} \\ b &= 90^\circ - \text{latitude of A} - \mathrm{P} \end{aligned}$$

where $\mathrm{P} = \tfrac{1}{2}(a_1 - b_1)\left(\frac{\mathrm{K} - 1}{\mathrm{K}}\right)$, and A is the farther station from the pole.

Problem (3).—Given the "latitudes" of A and B and the "azimuth" of B from A, required the "difference of longitude" and the "distance" AB. In this case we know two sides a and b, and the angle opposite one of them A, but before solving the triangle a small correction (P) must be applied to the values of the "co-latitudes" as follows:—

From Table II. take out the natural number "K" corresponding to the *mean* latitude of AB, then—

$$P = \frac{d}{2} \cdot \frac{K - 1}{K}$$

where $d =$ "difference in seconds" between "latitudes" of A and B,

$$\text{and } a = 90° - \text{latitude of B} + P$$
$$b = 90° - \text{latitude of A} - P$$

The triangle can be solved by such an artifice as described on p. 181.

Example of Problem (3).—

$$\text{At station A and B, given lat. of A} = 12°$$
$$\text{Lat. of B} = 12° \ 9' \ 15''{\cdot}37$$
$$\text{Azimuth of B from A} = 45°$$

Required the "diff. of long." between A and B, and the "distance" AB.

Now "K" or Nat. No. for lat. 12° 5′ from Table II. $= 1{\cdot}00653$

$$\therefore \ P = \frac{9' \ 15''{\cdot}37}{2} \times \frac{0{\cdot}00653}{1{\cdot}00653} = 1''{\cdot}8015$$

$$\therefore \ a = 90° - 12° \ 9' \ 15'' \ 37 + 1{\cdot}8'' = 77° \ 50' \ 46''{\cdot}43$$

and

$$b = 90° - 12° - 1''{\cdot}8 = 77° \ 59' \ 58''{\cdot}2$$

Put $\cos b = k \cos \theta$,
$\sin b \cos A = k \sin \theta$.

Then $\quad \tan \theta = \tan b \cos A$, where $A = 45°$ in this case
and $\quad k = \cos b \sec \theta$, whence θ and k are known
Now

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$= k \cos \theta \cos c + k \sin \theta \sin c$$
$$= k \cos (\theta - c), \text{ whence } c \text{ can be found}$$

In this case $c = 13' \ 0{\cdot}6''$, and can be reduced to feet to find the distance AB.

Then

$$\sin C = \frac{\sin A}{\sin a} \times \sin c$$

whence

$$C = 9' \ 24{\cdot}6'' = \text{diff. of longitude}$$

Examples for Exercise.

(1) Assuming the earth spherical, if λ be the latitude of the standard parallel in a simple conical projection, and λ_1 be any other latitude (both in circular measure), show that the lengths of these parallels in the projection will be in the ratio of $\{\cot \lambda - (\lambda_1 - \lambda)\}$ to $\cot \lambda$, whilst the true ratio is that of $\cos \lambda_1$ to $\cos \lambda$.

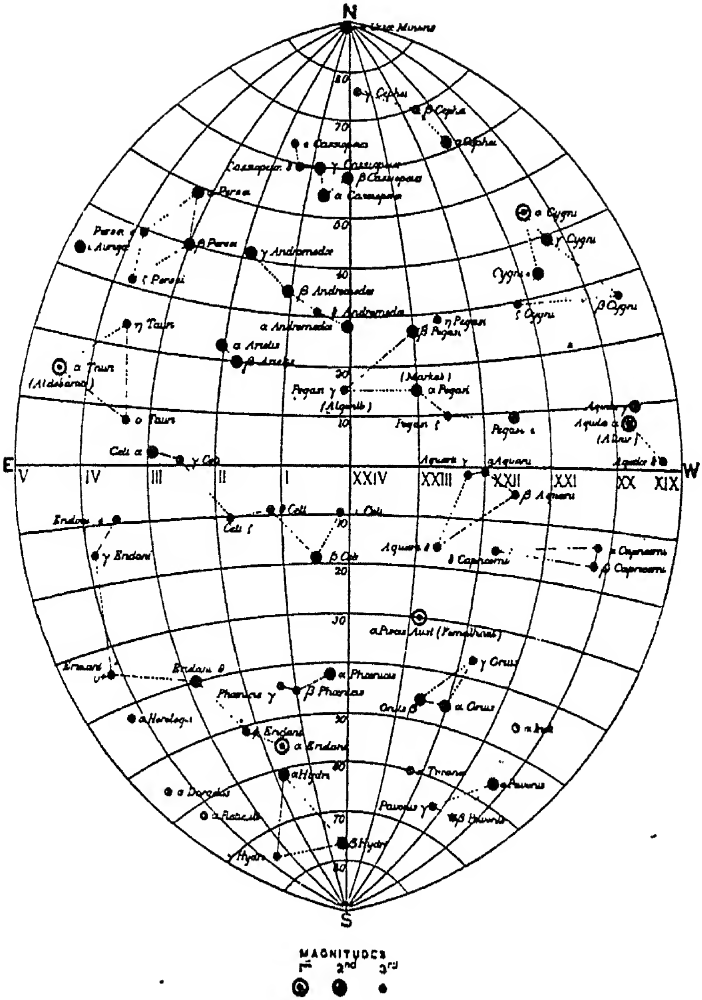
Hence show that if the standard parallel be 40° N., the increase in scale

along the parallels of 20° N. and 60° N. is about 5 per cent. and 8 per cent. respectively.

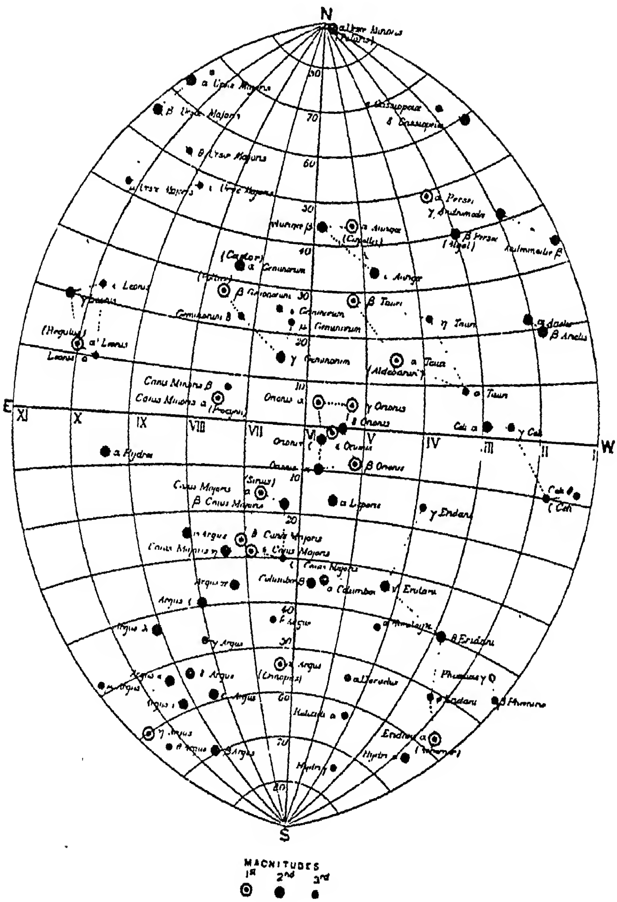
(2) In the simple conical with two standards at 20° and 40° respectively, show that the scale along the 30° parallel is too small by 1·5 per cent., and that along the 50° parallel is too great by about 5·5 per cent. The earth is assumed spherical. (*Note.*—The ratio of the radius for the 30° parallel to that for 20° is $\frac{\cos 20^\circ + \cos 40^\circ}{2 \cos 20^\circ}$; hence the lengths on the projection are also in this ratio. But they should be in the ratio of cos 30° to cos 20°.)

APPENDIX A

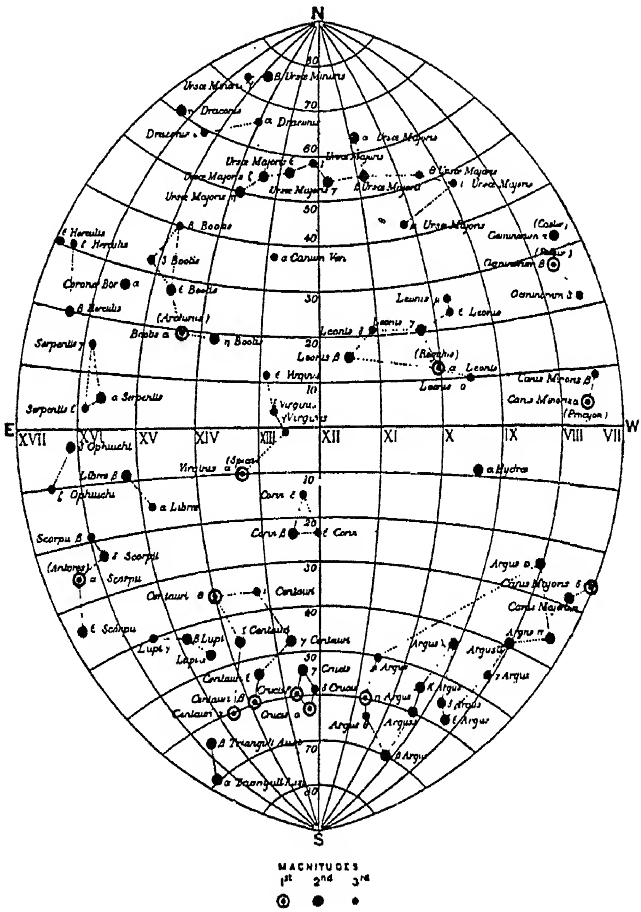
Star Chart I



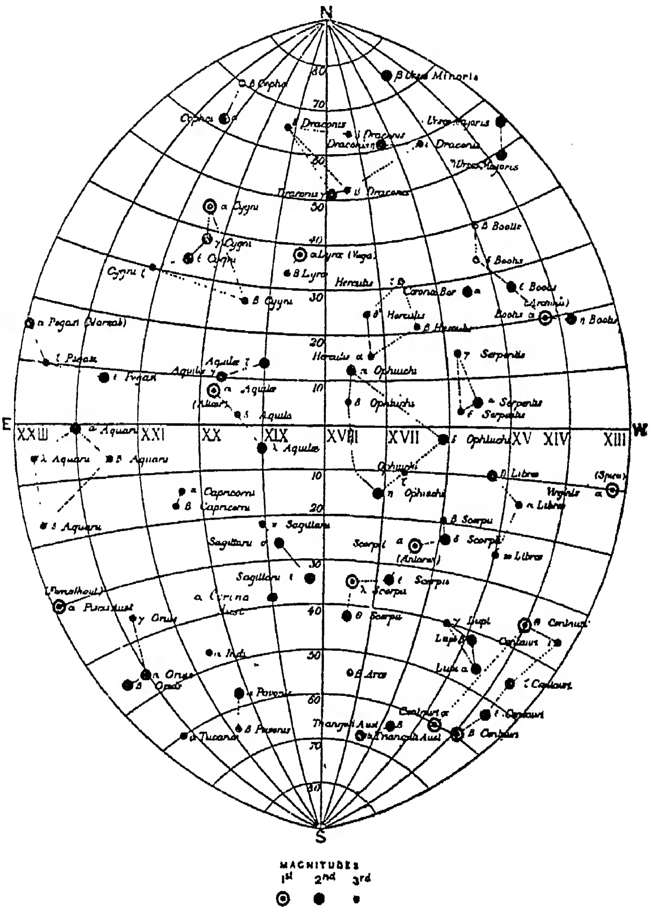
Star Chart II



Star Chart III



Star Chart IV



APPENDIX B

TABLE I

Mean Refraction

Barometer 30 in. Thermometer 50°

Appt. altitude. ° ′	Refraction. ′ ″
15 0	3 34·1
5	3 32·9
10	3 31·7
15	3 30·5
20	3 29·4
25	3 28·2
30	3 27·1
35	3 25·9
40	3 24·8
45	3 23·7
50	3 22·6
55	3 21·5
16 0	3 20·5
5	3 19·4
10	3 18·4
15	3 17·3
20	3 16·3
25	3 15·2
30	3 14·2
35	3 13·2
40	3 12·2
45	3 11·2
50	3 10·3
55	3 9·3
17 0	3 8·3
5	3 7·3
10	3 6·4
15	3 5·5
20	3 4·6
25	3 3·7
30	3 2·8
35	3 1·9
40	3 1·0
45	3 0·1

Appt. altitude. ° ′	Refraction. ′ ″
22 0	2 23·3
10	2 22·1
20	2 20·9
30	2 19·8
40	2 18·7
50	2 17·5
23 0	2 16·4
10	2 15·4
20	2 14·3
30	2 13·3
40	2 12·2
50	2 11·2
24 0	2 10·2
10	2 9·2
20	2 8·2
30	2 7·2
40	2 6·2
50	2 5·3
25 0	2 4·4
10	2 3·4
20	2 2·5
30	2 1·6
40	2 0·7
50	1 59·8
26 0	1 58·9
10	1 58·1
20	1 57·2
30	1 56·4
40	1 55·5
50	1 54·7
27 0	1 53·9
10	1 53·1
20	1 52·3
30	1 51·5
40	1 50·7

Appt. altitude. ° ′	Refraction. ′ ″
36 0	1 20·1
20	1 19·1
40	1 18·2
37 0	1 17·2
20	1 16·3
40	1 15·4
38 0	1 14·5
20	1 13·6
40	1 12·7
39 0	1 11·9
20	1 11·0
40	1 10·2
40 0	1 9·4
20	1 8·6
40	1 7·8
41 0	1 7·0
20	1 6·2
40	1 5·4
42 0	1 4·7
20	1 3·9
40	1 3·2
43 0	1 2·4
20	1 1·7
40	1 1·0
44 0	1 0·3
20	0 59·6
40	0 58·9
45 0	0 58·2
20	0 57·6
40	0 56·9
46 0	0 56·2
20	0 55·6
40	0 55·0
47 0	0 54·3
20	0 53·7

Appt. altitude. ° ′	Refraction. ″
64 0	28·4
30	27·8
65 0	27·2
30	26·5
66 0	25·9
30	25·3
67 0	24·7
30	24·1
68 0	23·6
30	23·0
69 0	22·4
30	21·8
70 0	21·2
30	20·6
71 0	20·1
30	19·5
72 0	18·9
30	18·3
73 0	17·8
30	17·2
74 0	16·7
30	16·1
75 0	15·6
30	15·0
76 0	14·5
30	14·0
77 0	13·5

18 0	2 57·5	28 0	1 49·2	48 0	0 52·5	78 0	12·4
5	2 56·6	10	1 48·4	20	0 51·8	30	11·8
10	2 55·8	20	1 47·7	40	0 51·2	79 0	11·3
15	2 54·9	30	1 46·9	49 0	0 50·6	30	10·8
20	2 54·1	40	1 46·2	20	0 50·0	80 0	10·3
25	2 53·2	50	1 45·5	40	0 49·4	30	9·7
30	2 52·4	29 0	1 44·8	50 0	0 48·9	81 0	9·2
35	2 51·6	10	1 44·1	20	0 48·3	30	8·7
40	2 50·8	20	1 43·4	40	0 47·7	82 0	8·2
45	2 50·0	30	1 42·7	51 0	0 47·2	30	7·7
50	2 49·2	40	1 42·0	20	0 46·6	83 0	7·2
55	2 48·4	50	1 41·3	40	0 46·0	30	6·6
19 0	2 47·7	30 0	1 40·6	52 0	0 45·5	84 0	6·1
5	2 46·9	10	1 39·9	20	0 44·9	30	5·6
10	2 46·1	20	1 39·3	40	0 44·4	85 0	5·1
15	2 45·3	30	1 38·6	53 0	0 43·9	30	4·6
20	2 44·6	40	1 38·0	20	0 43·3	86 0	4·1
25	2 43·8	50	1 37·3	40	0 42·8	30	3·6
30	2 43·1	31 0	1 36·7	54 0	0 42·3	87 0	3·1
35	2 42·3	10	1 36·1	20	0 41·8	30	2·5
40	2 41·6	20	1 35·5	40	0 41·3	88 0	2·0
45	2 40·9	30	1 34·8	55 0	0 40·8	30	1·5
50	2 40·2	40	1 34·2	20	0 40·3	89 0	1·0
55	2 39·5	50	1 33·6	40	0 39·8	30	0·5
20 0	2 38·8	32 0	1 33·0	56 0	0 39·3	90 0	0·0
5	2 38·1	10	1 32·4	20	0 38·8		
10	2 37·4	20	1 31·8	40	0 38·3		
15	2 36·7	30	1 31·2	57 0	0 37·8		
20	2 36·0	40	1 30·7	20	0 37·3		
25	2 35·3	50	1 30·1	40	0 36·8		
30	2 34·6	33 0	1 29·5	58 0	0 36·4		
35	2 33·9	10	1 28·9	20	0 35·9		
40	2 33·3	20	1 28·4	40	0 35·4		
45	2 32·6	30	1 27·8	59 0	0 35·0		
50	2 32·0	40	1 27·3	20	0 34·5		
55	2 31·3	50	1 26·7	40	0 34·0		
21 0	2 30·7	34 0	1 26·2	60 0	0 33·6		
5	2 30·0	10	1 25·6	20	0 33·1		
10	2 29·4	20	1 25·1	40	0 32·7		
15	2 28·7	30	1 24·6	61 0	0 32·3		
20	2 28·1	40	1 24·1	20	0 31·8		
25	2 27·5	50	1 23·6	40	0 31·4		
30	2 26·9	35 0	1 23·1	62 0	0 31·0		
35	2 26·3	10	1 22·5	20	0 30·5		
40	2 25·7	20	1 22·0	40	0 30·1		
45	2 25·1	80	1 21·5	63 0	0 29·7		
50	2 24·5	40	1 21·0	20	0 29·2		
55	2 23·9	50	1 20·5	40	0 28·8		

TABLE II
Refraction

Below 30 in.	*Mean Refraction.*														Above 30 in.
Subtract.	0′		1′		2′		3′		4′		5′		6′		Add.
Barometer.	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	Barometer.
	″	″	″	″	″	″	″	″	″	″	″	″	″	″	
23·00	0	7	14	21	28	35	42	49	56	63	70	77	84	91	
·2	0	7	14	20	27	34	41	48	55	61	68	75	81	88	
·4	0	7	13	20	26	33	40	46	53	59	66	73	79	86	
·6	0	7	13	19	26	32	38	45	51	57	64	70	77	83	
·8	0	6	12	19	25	31	37	43	50	56	62	68	74	81	
24·00	0	6	12	18	24	30	36	42	48	54	60	66	72	78	
·2	0	6	12	17	23	29	35	41	46	52	58	64	69	75	
·4	0	6	11	17	22	28	34	39	45	50	56	62	67	73	
·6	0	5	11	16	22	27	32	38	43	48	54	60	65	70	
·8	0	5	10	16	21	26	31	36	42	47	52	57	62	68	
25·00	0	5	10	15	20	25	30	35	40	45	50	55	60	65	
·2	0	5	10	14	19	24	29	34	38	43	48	53	57	62	
·4	0	5	9	14	18	23	28	32	37	41	46	51	55	60	
·6	0	4	9	13	18	22	26	31	35	39	44	49	53	57	
·8	0	4	8	13	17	21	25	29	34	38	42	46	50	55	
26·00	0	4	8	12	16	20	24	28	32	36	40	44	48	52	
·2	0	4	8	11	15	19	23	27	30	34	38	42	46	49	
·4	0	4	7	11	14	18	22	25	29	32	36	40	43	47	
·6	0	3	7	10	14	17	20	24	27	31	34	37	41	44	
·8	0	3	6	10	13	16	19	22	26	29	32	35	38	42	
27·00	0	3	6	9	12	15	18	21	24	27	30	33	36	39	
·2	0	3	6	9	11	14	17	20	22	25	28	31	34	37	
·4	0	3	5	8	10	13	16	18	21	23	26	29	31	34	
27·50	0	2	5	7	10	12	15	17	20	23	25	28	30	33	
·55	0	2	5	7	10	12	15	17	20	22	25	27	30	32	
·60	0	2	5	7	10	12	14	17	19	22	24	27	29	31	
·65	0	2	5	7	9	12	14	16	19	21	24	26	28	31	
·70	0	2	5	7	9	11	14	16	18	21	23	25	28	30	
27·75	0	2	4	7	9	11	13	16	18	20	23	25	27	29	
·80	0	2	4	7	9	11	13	15	18	20	22	24	27	29	
·85	0	2	4	6	9	11	13	15	17	19	22	24	26	28	
·90	0	2	4	6	8	10	13	15	17	19	21	23	25	27	
·95	0	2	4	6	8	10	12	14	16	18	21	23	25	27	
28·00	0	2	4	6	8	10	12	14	16	18	20	22	24	26	
·05	0	2	4	6	8	10	12	14	16	18	20	22	24	25	
·10	0	2	4	6	8	9	11	13	15	17	19	21	23	25	
·15	0	2	4	6	7	9	11	13	15	17	19	20	22	24	
·20	0	2	4	5	7	9	11	13	14	16	18	20	22	24	
28·25	0	2	3	5	7	8	10	12	14	16	18	19	21	23	
·30	0	2	3	5	7	8	10	12	14	15	17	19	21	22	

Subtract.	0′ 0″	0′ 30″	1′ 0″	1′ 30″	2′ 0″	2′ 30″	3′ 0″	3′ 30″	4′ 0″	4′ 30″	5′ 0″	5′ 30″	6′ 0″	6′ 30″	Add.
28·00	0	2	4	6	8	10	12	14	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
·05	0	2	4	6	8	10	12	14	16	18	20	22	24	25	
·10	0	2	4	6	8	9	11	13	15	17	19	21	23	25	
·15	0	2	4	6	7	9	11	13	15	17	19	20	22	24	
·20	0	2	4	5	7	9	11	13	14	16	18	20	22	24	
28·25	0	2	3	5	7	9	10	12	14	16	18	19	21	23	
·30	0	2	3	5	7	8	10	12	14	15	17	19	21	22	
·35	0	2	3	5	7	8	10	12	13	15	17	18	20	22	
·40	0	2	3	5	6	8	10	11	13	14	16	18	19	21	
·45	0	2	3	5	6	8	9	11	12	14	16	17	19	20	
28·50	0	1	3	4	6	7	9	10	12	14	15	17	18	20	31·50
·55	0	1	3	4	6	7	9	10	12	13	15	16	17	19	·45
·60	0	1	3	4	6	7	8	10	11	13	14	15	17	18	·40
·65	0	1	3	4	5	7	8	9	11	12	14	15	16	18	·35
·70	0	1	3	4	5	6	8	9	10	12	13	14	16	17	·30
28·75	0	1	2	4	5	6	7	9	10	11	13	14	15	16	31·25
·80	0	1	2	4	5	6	7	8	10	11	12	13	14	16	·20
·85	0	1	2	3	5	6	7	8	9	10	12	13	14	15	·15
·90	0	1	2	3	4	5	7	8	9	10	11	12	13	14	·10
·95	0	1	2	3	4	5	6	7	8	9	11	12	13	14	·05
29·00	0	1	2	3	4	5	6	7	8	9	10	11	12	13	31·00
·05	0	1	2	3	4	5	6	7	8	9	10	11	11	12	30·95
·10	0	1	2	3	4	4	5	6	7	8	9	10	11	12	·90
·15	0	1	2	3	3	4	5	6	7	8	9	9	10	11	·85
·20	0	1	2	2	3	4	5	6	6	7	8	9	10	10	·80
29·25	0	1	1	2	3	4	4	5	6	7	8	8	9	10	30·75
·30	0	1	1	2	3	3	4	5	6	6	7	8	8	9	·70
·35	0	1	1	2	3	3	4	5	5	6	7	7	8	9	·65
·40	0	1	1	2	2	3	4	4	5	5	6	7	7	8	·60
·45	0	1	1	2	2	3	3	4	4	5	6	6	7	7	·55
29·50	0	0	1	1	2	2	3	3	4	5	5	6	6	7	30·50
·55	0	0	1	1	2	2	3	3	4	4	5	5	5	6	·45
·60	0	0	1	1	2	2	2	3	3	4	4	4	5	5	·40
·65	0	0	1	1	1	2	2	2	3	3	4	4	4	5	·35
·70	0	0	1	1	1	1	2	2	2	3	3	3	4	4	·30
29·75	0	0	0	1	1	1	1	2	2	2	3	3	3	3	30·25
·80	0	0	0	1	1	1	1	1	2	2	2	2	2	3	·20
·85	0	0	0	0	1	1	1	1	1	1	2	2	2	2	·15
·90	0	0	0	0	0	0	1	1	1	1	1	1	1	1	·10
·95	0	0	0	0	0	0	0	0	0	0	1	1	1	1	·05
30·00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30·00

MEAN REFRACTION.

NOTE.—Should the barometer stand at *less* than 23 inches the barometer correction can be computed from the following quantities, which are made out for *each inch* of the barometer below 30 inches, and which are dependent on the altitudes.

Altitude	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	30°	35°	40°	45°	50°	55°	60°	70°	80°
Barometer correction for each 1″ below 30″	10·8″	9·8″	9·0″	8·3″	7·7″	7·18″	6·73″	6·31″	5·98″	5·61″	5·31″	5·04″	4·79″	4·57″	4·35″	4·16″	3·97″	3·81″	3·65″	3·50″	3·36″	2·78″	2·32″	1·94″	1·63″	1·36″	1·12″	0·71″	0·34″

TABLE III.—*continued*

CORRECTION OF THE MEAN REFRACTION FOR HEIGHT OF THERMOMETER

Thermo. SUB-TRACT.	Mean Refraction. 0′		1′		2′		3′		4′		5′		6′		7′		Thermo. SUB-TRACT.
	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	
°	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	°
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	50
51	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	51
52	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	52
53	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	3	53
54	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	54
55	0	0	1	1	1	1	2	2	2	3	3	3	4	4	4	5	55
56	0	0	1	1	1	2	2	2	3	3	4	4	4	5	5	6	56
57	0	0	1	1	2	2	2	3	3	4	4	5	5	6	6	6	57
58	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	58
59	0	1	1	2	2	3	3	4	4	5	5	6	6	7	8	8	59
60	0	1	1	2	2	3	3	4	5	5	6	7	7	8	9	9	60
61	0	1	1	2	3	3	4	4	5	6	7	7	8	9	9	10	61
62	0	1	1	2	3	3	4	5	6	6	7	8	9	9	10	11	62
63	0	1	1	2	3	4	5	5	6	7	8	8	9	10	11	12	63
64	0	1	2	2	3	4	5	6	7	7	8	9	10	11	12	13	64
65	0	1	2	3	3	4	5	6	7	8	9	10	11	12	13	14	65
66	0	1	2	3	4	5	6	6	7	8	9	10	11	12	14	15	66
67	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	67
68	0	1	2	3	4	5	6	7	8	9	11	11	13	14	15	16	68
69	0	1	2	3	4	5	7	8	9	10	11	12	13	15	16	17	69
70	0	1	2	3	5	6	7	8	9	10	12	12	14	16	17	18	70
71	0	1	2	4	5	6	7	8	10	11	12	13	15	16	18	19	71
72	0	1	2	4	5	6	8	9	10	11	13	14	16	17	18	20	72
73	0	1	3	4	5	7	8	9	11	12	13	14	16	18	19	21	73
74	0	1	3	4	5	7	8	10	11	12	14	15	17	18	20	22	74
75	0	1	3	4	6	7	8	10	11	13	14	16	18	19	21	22	75
76	0	1	3	4	6	7	9	10	12	13	15	16	18	20	22	23	76
77	0	1	3	5	6	8	9	11	12	14	16	17	19	21	22	24	77
78	0	2	3	5	6	8	9	11	13	14	16	18	20	21	23	25	78
79	0	2	3	5	6	8	10	11	13	15	17	18	20	22	24	26	79
80	0	2	3	5	7	8	10	12	14	15	17	19	21	23	25	27	80
81	0	2	3	5	7	9	10	12	14	16	18	20	21	24	26	28	81
82	0	2	4	5	7	9	11	13	14	16	18	20	22	24	26	28	82
83	0	2	4	5	7	9	11	13	15	17	19	21	23	25	27	29	83
84	0	2	4	6	8	9	11	13	15	17	19	21	23	26	28	30	84
85	0	2	4	6	8	10	12	14	16	18	20	22	24	26	29	31	85
86	0	2	4	6	8	10	12	14	16	18	20	23	25	27	29	32	86
87	0	2	4	6	8	10	12	14	17	19	21	23	25	28	30	32	87
88	0	2	4	6	8	10	13	15	17	19	21	24	26	28	31	33	88
89	0	2	4	6	9	11	13	15	17	20	22	24	27	29	32	34	89
90	0	2	4	7	9	11	13	16	18	20	23	25	27	30	32	35	90
91	0	2	4	7	9	11	14	16	18	21	23	25	28	31	33	36	91
92	0	2	5	7	9	11	14	16	19	21	24	26	29	31	34	37	92
93	0	2	5	7	9	12	14	17	19	22	24	27	29	32	35	37	93
94	0	2	5	7	10	12	14	17	19	22	25	27	30	33	35	38	94
95	0	2	5	7	10	12	15	17	20	22	25	28	30	33	36	39	95
96	0	2	5	7	10	12	15	18	20	23	26	28	31	34	37	40	96
97	0	3	5	8	10	13	15	18	21	23	26	29	32	35	38	41	97
98	0	3	5	8	10	13	16	18	21	24	27	29	32	35	38	41	98
99	0	3	5	8	11	13	16	19	21	24	27	30	33	36	39	42	99
100	0	3	5	8	11	13	16	19	22	25	28	31	34	37	40	43	100
SUB-TRACT.	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	SUB-TRACT.
Thermo.	0′		1′		2′		3′		4′		5′		6′		7′		Thermo.

MEAN REFRACTION.

TABLE III.—*continued*

Correction of the Mean Refraction for Height of Thermometer

Thermo. Subtract.	Mean Refraction.																Thermo. Subtract.
	0′		1′		2′		3′		4′		5′		6′		7′		
	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	
°	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	°
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	50
51	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	51
52	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	52
53	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	3	53
54	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	54
55	0	0	1	1	1	1	2	2	2	3	3	3	4	4	4	5	55
56	0	0	1	1	1	2	2	2	3	3	4	4	4	5	5	6	56
57	0	0	1	1	2	2	2	3	3	4	4	5	5	6	6	6	57
58	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	58
59	0	1	1	2	2	3	3	4	4	5	5	6	6	7	8	8	59
60	0	1	1	2	2	3	3	4	5	5	6	7	7	8	9	9	60
61	0	1	1	2	3	3	4	4	5	6	7	7	8	9	9	10	61
62	0	1	1	2	3	3	4	5	6	6	7	8	9	9	10	11	62
63	0	1	1	2	3	4	5	5	6	7	8	8	9	10	11	12	63
64	0	1	2	2	3	4	5	6	7	7	8	9	10	11	12	13	64
65	0	1	2	3	3	4	5	6	7	8	9	10	11	12	13	14	65
66	0	1	2	3	4	5	6	6	7	8	9	10	11	12	14	15	66
67	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	67
68	0	1	2	3	4	5	6	7	8	9	11	11	13	14	15	16	68
69	0	1	2	3	4	5	7	8	9	10	11	12	13	15	16	17	69
70	0	1	2	3	5	6	7	8	9	10	12	12	14	16	17	18	70
71	0	1	2	4	5	6	7	8	10	11	12	13	15	16	18	19	71
72	0	1	2	4	5	6	8	9	10	11	13	14	16	17	18	20	72
73	0	1	3	4	5	7	8	9	11	12	13	14	16	18	19	21	73
74	0	1	3	4	5	7	8	10	11	12	14	15	17	18	20	22	74
75	0	1	3	4	6	7	8	10	11	13	14	16	18	19	21	22	75
76	0	1	3	4	6	7	9	10	12	13	15	16	18	20	22	23	76
77	0	1	3	5	6	8	9	11	12	14	16	17	19	21	22	24	77
78	0	2	3	5	6	8	9	11	13	14	16	18	20	21	23	25	78
79	0	2	3	5	6	8	10	11	13	15	17	18	20	22	24	26	79
80	0	2	3	5	7	8	10	12	14	15	17	19	21	23	25	27	80
81	0	2	3	5	7	9	10	12	14	16	18	20	21	24	26	28	81
82	0	2	4	5	7	9	11	13	14	16	18	20	22	24	26	28	82
83	0	2	4	5	7	9	11	13	15	17	19	21	23	25	27	29	83
84	0	2	4	6	8	9	11	13	15	17	19	21	23	26	28	30	84
85	0	2	4	6	8	10	12	14	16	18	20	22	24	26	29	31	85
86	0	2	4	6	8	10	12	14	16	18	20	23	25	27	29	32	86
87	0	2	4	6	8	10	12	14	17	19	21	23	25	28	30	32	87
88	0	2	4	6	8	10	13	15	17	19	21	24	26	28	31	33	88
89	0	2	4	6	9	11	13	15	17	20	22	24	27	29	32	34	89
90	0	2	4	7	9	11	13	16	18	20	23	25	27	30	32	35	90
91	0	2	4	7	9	11	14	16	18	21	23	25	28	31	33	36	91
92	0	2	5	7	9	11	14	16	19	21	24	26	29	31	34	37	92
93	0	2	5	7	9	12	14	17	19	22	24	27	29	32	35	37	93
94	0	2	5	7	10	12	14	17	19	22	25	27	30	33	35	38	94
95	0	2	5	7	10	12	15	17	20	22	25	28	30	33	36	39	95
96	0	2	5	7	10	12	15	18	20	23	26	28	31	34	37	40	96
97	0	3	5	8	10	13	15	18	21	23	26	29	32	35	38	41	97
98	0	3	5	8	10	13	16	18	21	24	27	29	32	35	38	41	98
99	0	3	5	8	11	13	16	19	21	24	27	30	33	36	39	42	99
100	0	3	5	8	11	13	16	19	22	25	28	31	34	37	40	43	100
Subtract. Thermo.	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	Subtract. Thermo.
	0′		1′		2′		3′		4′		5′		6′		7′		
	Mean Refraction.																

TABLE IV

SUN'S PARALLAX IN ALTITUDE.

Date.	Altitude.										Date.
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	
	"	"	"	"	"	"	"	"	"	"	
January 1st ...	9·00	8·86	8·46	7·79	6·89	5·79	4·50	3·08	1·57	0	
February 1st ..	8·98	8·85	8·44	7·78	6·88	5·77	4·49	3·07	1·56	0	December 1st.
March 1st ...	8·92	8·79	8·38	7·72	6·83	5·74	4·46	3·05	1·55	0	November 1st.
April 1st ...	8·85	8·72	8·32	7·66	6·78	5·69	4·42	3·03	1·54	0	October 1st.
May 1st ...	8·78	8·65	8·25	7·60	6·73	5·65	4·39	3·00	1·53	0	September 1st.
June 1st ...	8·72	8·59	8·20	7·55	6·68	5·61	4·36	2·98	1·52	0	August 1st.
July 1st ...	8·70	8·57	8·18	7·53	6·66	5·59	4·35	2·97	1·51	0	

TABLE V

Reduction of Circum-Meridian Altitudes to the Meridian

$$m = \frac{2 \sin^2 \frac{1}{2}t}{\sin 1''}$$

t	m.	1 m.	2 m.	3 m.	4 m.	5 m.	6 m.	7 m.	8 m.	9 m.	10 m.	11 m.	12 m.	13 m.	14 m.	15 m.	16 m.	17 m.	18 m.	19 m.
	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
0	0·0	2·0	7·8	17·7	31·4	49·1	70·7	96·2	125·7	159·0	196·3	237·5	382·7	331·7	384·7	441·6	502·5	567·2	635·9	708·4
2	0·0	2·1	8·1	18·1	31·9	49·7	71·5	97·1	126·7	160·2	197·6	239·0	284·3	333·4	386·6	443·6	504·6	569·4	638·2	710·9
4	0·0	2·2	8·4	18·5	32·5	50·4	72·3	98·0	127·8	161·4	198·9	240·4	285·8	335·2	388·4	445·6	506·7	571·6	640·6	713·4
6	0·0	2·4	8·7	18·9	33·0	51·1	73·1	99·0	128·8	162·6	200·3	241·9	287·4	336·8	390·2	447·5	508·8	573·9	642·9	715·9
8	0·0	2·5	8·9	19·3	33·5	51·7	73·9	99·9	129·9	163·8	201·6	243·3	289·0	338·6	392·1	449·5	510·9	576·1	645·3	718·4
10	0·1	2·7	9·2	19·7	34·1	52·4	74·7	100·8	130·9	165·0	202·9	244·8	290·6	340·3	393·9	451·5	513·0	578·4	647·7	720.9
12	0·1	2·8	9·5	20·1	34·6	53·1	75·5	101·8	132·0	166·2	204·2	246·3	292·2	342·0	395·8	453·5	515·1	580·6	650·0	723·4
14	0·1	3·0	9·8	20·5	35·2	53·8	76·3	102·7	133·1	167·4	205·6	247·7	293·8	343·7	397·6	455·5	517·2	582·9	652·4	725·9
16	0·1	3·1	10·1	20·9	35·7	54·5	77·1	103·7	134·2	168·6	206·9	249·2	295·4	345·5	399·5	457·5	519·3	585·1	654·8	728·4
18	0·2	3·3	10·4	21·4	36·3	55·1	77·9	104·6	135·3	169·8	208·3	250·7	297·0	347·2	401·4	459·5	521·5	587·4	657·2	730·9
20	0·2	3·5	10·7	21·8	36·9	55·8	78·8	105·6	136·3	171·0	209·6	252·2	298·6	349·0	403·3	461·5	523·6	589·6	659·6	733·5
22	0·3	3·7	11·0	22·3	37·4	56·5	79·6	106·6	137·4	172·2	211·0	253·6	300·2	350·7	405·1	463·5	525·7	591·9	662·0	736·0
24	0·3	3·8	11·3	22·7	38·0	57·3	80·4	107·5	138·5	173·5	212·3	255·1	301·8	352·5	407·0	465·5	527·9	594·2	664·4	738·5
26	0·4	4·0	11·6	23·1	38·6	58·0	81·3	108·5	139·6	174·7	213·7	256·6	303·5	354·2	408·9	467·5	530·0	596·5	666·8	741·1
28	0·4	4·2	11·9	23·6	39·2	58·7	82·1	109·5	140·7	175·9	215·1	258·1	305·1	356·0	410·8	469·5	532·2	598·7	669·2	743·6
30	0·5	4·4	12·3	24·0	39·8	59·4	83·0	110·4	141·8	177·2	216·4	259·6	306·7	357·7	412·7	471·5	534·3	601·0	671·6	746·2
32	0·6	4·6	12·6	24·5	40·3	60·1	83·8	111·4	143·0	178·4	217·8	261·1	308·4	359·5	414·6	473·6	536·5	603·3	674·1	748·7
34	0·6	4·8	12·9	25·0	40·9	60·8	84·7	112·4	144·1	179·7	219·2	262·6	310·0	361·1	416·5	475·6	538·7	605·6	676·5	751·3
36	0·7	5·0	13·3	25·4	41·5	61·6	85·5	113·4	145·2	180·9	220·6	264·1	311·6	363·1	418·4	477·6	540·8	607·9	678·9	753·8
38	0·8	5·2	13·6	25·9	42·1	62·3	86·4	114·4	146·3	182·2	222·0	265·7	313·3	364·8	420·3	479·7	543·0	610·2	681·3	756·4
40	0·9	5·4	14·0	26·4	42·8	63·0	87·3	115·4	147·5	183·5	223·4	267·2	315·0	366·6	422·2	481·7	545·2	612·5	683·8	759·0
42	1·0	5·7	14·3	26·9	43·4	63·8	88·1	116·4	148·6	184·7	224·8	268·7	316·6	368·4	424·2	483·8	547·4	614·8	686·2	761·5
44	1·1	5·9	14·7	27·4	44·0	64·5	89·0	117·4	149·7	186·0	226·2	270·3	318·3	370·2	426·1	485·8	549·5	617·2	688·7	764·1
46	1·2	6·1	15·0	27·9	44·3	65·3	89·9	118·4	150·9	187·3	227·6	271·8	319·9	372·0	428·0	487·9	551·7	619·5	691·1	766·7
48	1·3	6·4	15·4	28·3	45·2	66·0	90·8	119·5	152·0	188·5	229·0	273·3	321·6	373·8	429·9	490·0	553·0	621·8	693·6	769·3
50	1·4	6·6	15·8	28·8	45·9	66·8	91·7	120·5	153·2	189·8	230·4	274·9	323·3	375·6	431·9	492·0	556·1	624·1	696·0	771·9
52	1·5	6·8	16·1	29·4	46·5	67·6	92·6	121·5	154·3	191·1	231·8	276·4	325·0	377·4	433·8	494·1	558·3	626·5	698·5	774·5
54	1·6	7·1	16·5	29·0	47·1	68·3	93·5	122·5	155·5	192·4	233·2	278·0	326·7	379·3	435·8	496·2	560·5	628·8	701·0	777·1
56	1·7	7·3	16·9	30·4	47·8	69·1	94·4	123·6	156·7	193·7	234·7	279·5	328·4	381·1	437·7	498·3	562·8	631·2	703·5	779·7
58	1·8	7·6	17·3	30·9	48·4	69·9	95·3	124·6	157·8	195·0	236·1	281·1	330·0	382·9	439·7	500·4	565·0	633·5	705·9	782·3

APPENDIX C

TABLE I

Table for reducing "Length in Feet" to "Seconds of Contained Arc"

Lat.	Geodetic distance in feet. 10,000	20,000	30,000	40,000	50,000	60,000	70,000	80,000	90,000
°	″	″	″	″	″	″	″	″	″
0	93·57	197·13	295·70	394·27	492·84	591·40	689·97	788·54	887·10
2	·57	·13	·70	·27	·83	·40	·97	·53	·10
4	·57	·13	·70	·26	·83	·39	·96	·52	·09
6	·56	·13	·69	·25	·82	·38	·94	·51	·07
8	·56	·12	·68	·24	·80	·36	·92	·49	·04
10	·56	·11	·67	·23	·78	·34	·90	·46	·01
12	98·55	197·10	295·66	394·21	492·76	591·32	689·87	788·42	886·97
14	·55	·09	·64	·19	·74	·28	·83	·38	·93
16	·54	·08	·62	·17	·71	·25	·79	·33	·88
18	·54	·07	60	·14	·68	·21	·75	·28	·82
20	·53	·06	·58	·11	·64	·17	·70	·22	·75
22	98·52	197·04	295·56	394·08	492·60	591·12	689·64	788·16	886·68
24	·51	·02	·54	·04	·56	·07	·58	·09	·61
26	·50	·01	·51	·01	·51	·02	·52	·02	·52
28	·49	196·99	·48	393·97	·47	590·96	·45	787·95	·44
30	·48	·97	·45	·93	·42	·90	·38	·87	·35
32	98·47	196 95	295·42	393·89	492·37	590·84	689·31	787·78	886·26
34	·46	·92	·39	·85	·31	·77	·24	·70	·16
36	·45	·90	·35	·81	·26	·71	·16	·61	·06
38	·44	·88	·32	·76	·20	·64	·08	·52	885·96
40	·43	·86	·29	·72	·14	·57	·00	·43	·86
42	98·42	196·83	295·25	393·67	492·09	590·50	688·92	787·34	885·75
44	·41	·81	·22	·62	·03	·43	·84	·24	·65
46	·39	·79	·18	·58	491·97	·36	·76	·15	·54
48	·38	·76	·15	·53	·91	·29	·68	·06	·44
50	·37	·74	·11	·48	·85	·22	·59	786·96	·34
52	98·36	196·72	295·08	393 44	491·79	590·15	688·51	786·86	885·23
54	·35	·70	·04	·39	·74	·09	·44	·78	·13
56	·34	·67	·01	·35	·68	·02	·36	·70	·03
58	·33	·65	294·98	·30	·63	589·96	·28	·61	884·94
60	·32	·63	·95	·26	·58	·90	·21	·53	·84

Computed from formula—

$$c'' = \frac{\text{length in feet} \times 180 \times 360}{}$$

where ν = normal to the meridian terminated by the minor axis.

Calculated on the following constants:—

Major axis = 41,852,696 feet

Minor axis = 41,710,466 feet

TABLE II

VALUES OF QUANTITY* TO BE APPLIED FOR "CORRECTION OF SPHERICALLY COMPUTED DIFFERENCE OF LATITUDE." [*Vide* PROBLEMS (1), (2) AND (3).]

Lat.	Log.	Natural number K.	Lat.	Log.	Natural number K.
°			°		
0	0·0029568	1·00683	30	0·0022195	1·00512
1	9559	83	31	1744	02
2	9532	82	32	1285	1·00491
3	9487	81	33	0818	81
4	9425	80	34	0344	70
5	9344	78	35	0·0019863	58
6	0·0029246	1·00676	36	0·0019375	1·00447
7	9130	73	37	8882	36
8	8997	70	38	8384	24
9	8847	66	39	7882	13
10	8679	63	40	7375	01
11	0·0028495	1·00658	41	0·0016866	1·00389
12	8294	54	42	6354	77
13	8077	49	43	5840	65
14	7843	43	44	5325	53
15	7594	37	45	4809	42
16	0·0027329	1·00631	46	0·0014293	1·00330
17	7048	25	47	3778	18
18	6753	18	48	3264	06
19	6444	11	49	2751	1·00294
20	6120	03	50	2241	82
21	0·0025782	1·00595	51	0·0011734	1·00271
22	5431	87	52	1231	59
23	5067	79	53	0732	47
24	4690	70	54	0238	36
25	4302	61	55	0·0009750	25
26	0·0023902	1·00552	56	0·0009267	1·00214
27	3490	42	57	8792	03
28	3068	33	58	8323	1·00192
29	2636	23	59	7863	81
30	2195	12	60	7411	71

* This quantity is $\frac{\nu}{\rho}$, where ν = normal to meridian terminated by the minor axis, and ρ = radius of curvature to the meridian.

APPENDIX D

The Greek Alphabet of 24 Letters

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zēta	Σ	σ	Sigma
H	η	Ēta	T	τ	Tau
Θ	θ	Thēta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

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P

R

S

(PR. 24)

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